CONDENSED
LESSON

3.1 Recursive Sequences

In this lesson you will

- find **recursive sequences** associated with toothpick patterns
- find missing values in recursive sequences
- write **recursive routines** that generate sequences

A **recursive sequence** is an ordered list of numbers generated by applying a rule to each successive number. For example, the sequence 100, 95, 90, 85, 80, 75, . . . is generated by applying the rule "subtract 5." Example A in your book shows how to use your calculator to generate a recursive sequence. Work through the example and make sure you understand it.

Investigation: Recursive Toothpick Patterns

Steps 1–4 Draw or use toothpicks to build the pattern of triangles on page 159 of your book, using one toothpick for each side of the smallest triangle. For each figure, find the total number of toothpicks and the number of toothpicks in the perimeter.

Build Figures 4–6 of the pattern. This table shows the number of toothpicks and the perimeter of each figure.

	Number of toothpicks	Perimeter
Figure 1	3	3
Figure 2	5	4
Figure 3	7	5
Figure 4	9	6
Figure 5	11	7
Figure 6	13	8

To find the number of toothpicks in a figure, add 2 to the number in the previous figure. To find the perimeter of a figure, add 1 to the perimeter of the previous figure. Below are the recursive routines to generate these number sequences on your calculator.

Number of toothpicks:	**Perimeter:**
Press 3 [ENTER].	Press 3 [ENTER].
Press $+2$.	Press $+1$.
Press [ENTER] to generate each successive term.	Press [ENTER] to generate each successive term.

Build Figure 10, and find the number of toothpicks and the perimeter. Use your calculator routines to check your counts. (The tenth time you press [ENTER], you will see the count for Figure 10.) There are 21 toothpicks in Figure 10 with 12 toothpicks on the perimeter.

(continued)

Steps 5–6 Repeat Steps 1–4 for a pattern of squares. Here is what the pattern should look like.

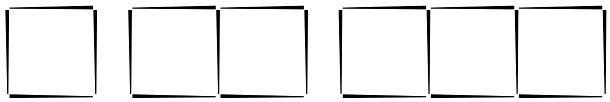


Figure 1 Figure 2 Figure 3

Look for rules for generating sequences for the number of toothpicks and the perimeter of each figure. You should find that the number of toothpicks in each figure is 3 more than the number in the previous figure and that the perimeter of each figure is 2 more than that of the previous perimeter. Notice that if you consider the length of a toothpick to be 1 unit, the area of Figure 1 is 1, the area of Figure 2 is 2, and so on.

Steps 7–8 Create your own pattern from toothpicks and, on your calculator, find recursive routines to produce the number sequences for the number of toothpicks, the perimeter, and the area.

Here is one pattern and the table and recursive routines that go with it.

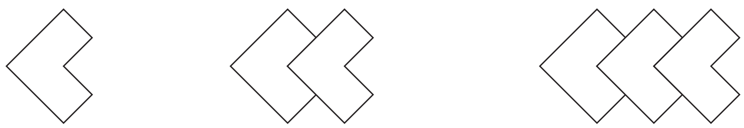


Figure 1 Figure 2 Figure 3

	Number of toothpicks	Perimeter	Area
Figure 1	8	8	3
Figure 2	14	12	6
Figure 3	20	16	9
Figure 4	26	20	12
Figure 12	74	52	36

Here are recursive routines that describe how the figures grow.

Number of toothpicks:	Perimeter:	Area:
Press 8 ENTER.	Press 8 ENTER.	Press 3 ENTER.
Press +6.	Press +4.	Press +3.
Press ENTER repeatedly.	Press ENTER repeatedly.	Press ENTER repeatedly.

For each routine, you can find the result for the figure with 40 puzzle pieces by pressing ENTER 40 times. You need 242 toothpicks to build the figure. The perimeter of the figure is 164, and the area is 120.

To find the number of pieces needed for a figure with area 150, use your area routine to generate numbers until you get to 150. You must press ENTER 50 times, so you would need 50 pieces. Now, use your number-of-toothpicks routine, pressing ENTER 50 times. The result is 302, so you need 302 toothpicks to build the figure.

Now, read Example B in your book, which gives you practice finding missing numbers in recursive sequences.

CONDENSED LESSON

3.2 Linear Plots

In this lesson you will

- use your calculator to apply several recursive routines at once
- graph values generated by recursive routines
- understand how the start value and rule of a recursive routine are reflected in the graph

Follow along with the example on pages 165–166 of your book and make sure you understand it.

Investigation: On the Road Again

Steps 1–3 In your book, read the introduction to the investigation and Step 1. You are given the speed of each vehicle in miles per hour. You can use dimensional analysis to convert each speed to miles per minute (mi/min). For example,

$$\frac{72 \text{ miles}}{1 \cancel{\text{hour}}} \cdot \frac{1 \cancel{\text{hour}}}{60 \text{ minutes}} = \frac{72 \text{ miles}}{60 \text{ minutes}} = 1.2 \text{ miles per minute}$$

Here are the speeds of the three vehicles in miles per minute.

minivan: 1.2 mi/min

pickup: 1.1 mi/min

sports car: 0.8 mi/min

Use these speeds to write recursive routines for finding each vehicle's distance from Flint after each minute.

The minivan starts 220 miles from Flint. After each minute, it is 1.2 miles closer to Flint. So the start value is 220, and the rule is "subtract 1.2."

The pickup starts 0 miles from Flint. After each minute, it is 1.1 miles farther from Flint. So the start value is 0, and the rule is "add 1.1."

The sports car starts 35 miles from Flint. After each minute, it is 0.8 mile farther from Flint. So the start value is 35, and the rule is "add 0.8."

To enter the recursive routines on your calculator, enter a list of start values, {220, 0, 35}. Then, apply the rules by entering

{Ans(1) − 1.2, Ans(2) + 1.1, Ans(3) + 0.8}

Use your calculator to find the distance from Flint each minute for the first few minutes. Record your results in a table. Then, change the rules to find distances at 10-minute intervals. To do this, multiply the numbers being added or subtracted by 10. Here are the new rules.

{Ans(1) − 12, Ans(2) + 11, Ans(3) + 8}

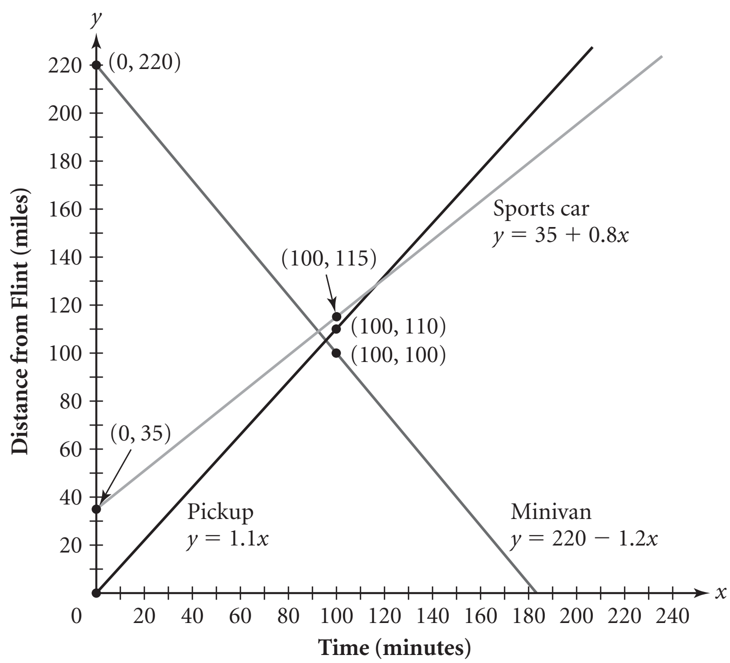
(continued)

Here is a table with a few values filled in. A complete table would have many more values and show time values until each vehicle reaches its destination.

Time (min)	Minivan (mi)	Sports car (mi)	Pickup (mi)
0	220	35	0
1	218.8	35.8	1.1
2	217.6	36.6	2.2
5	214	39	5.5
10	208	43	11
100	100	115	110

Steps 4–11 You can plot this information in a graph, with time on the x-axis and distance from Flint on the y-axis. Notice that the points for each vehicle fall on a line. It makes sense to connect the points to represent every possible instant of time.

The start value for each routine is the value where the graph crosses the y-axis. The recursive rule affects how much the distance value changes when the time value increases by 1. This determines the steepness of the line.



The line for the minivan slants down from left to right because the minivan's distance from Flint is decreasing with time. The lines for the other vehicles slant upward because their distances from Flint are increasing with time.

The lines for the minivan and the sports car cross at about (90, 110). This means that these vehicles pass each other after about 90 minutes, when they are both about 110 miles from Flint. At this time, the pickup is about 100 miles from Flint.

The line for the pickup is steeper than the line for the sports car, indicating that the pickup is traveling faster. The lines for the pickup and the sports car cross at about (115, 130), indicating that the pickup passes the sports car after about 115 minutes, when both vehicles are about 130 miles from Flint.

The line for the minivan crosses the x-axis before the lines for the other vehicles reach the 220-mile mark on the y-axis, indicating that the minivan reaches its destination first. The minivan reaches Flint in around 185 minutes. The pickup reaches the bridge in 200 minutes. The sports car reaches the bridge in about 230 minutes.

In this problem, we are assuming the vehicles travel at a constant speed, never stopping or slowing down. Realistically, the vehicles would change speeds, which would be indicated by changes in the steepness of the graph, and they would stop occasionally, which would be indicated by flat portions of the graph. You could not write one recursive routine to generate such graphs; you would have to write different routines for each interval with a different speed.

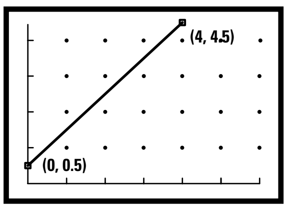
CONDENSED LESSON

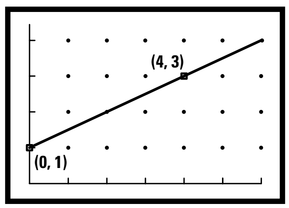
3.3 Time-Distance Relationships

In this lesson you will

- explore **time-distance relationships** using various walking scenarios
- examine how **starting position, speed, direction,** and **final position** influence a graph and an equation

Time-distance graphs provide a lot of information about the "walks" they picture. In the two walks represented below, the walker was moving away from the motion sensor at a steady rate. You can tell this because the distance is increasing, and the lines are straight. The first walker starts 0.5 meter from the sensor. Where does the second walker start? The first walker travels $4.5 - 0.5 = 4$ meters in $4 - 0 = 4$ seconds, or 1 meter per second (m/s). What is the second walker's rate?





The investigation practices interpreting walking graphs, and drawing graphs that fit a set of walking instructions.

Investigation: Walk the Line

Step 1 Look at graphs a–c on page 172 of your book. To write walking instructions for a graph, describe the start position, the direction of movement, and the speed.

For Graph a, the walker walks $4 - 2 = 2$ m in $6 - 0 = 6$ s, so the speed is $\frac{2\text{ m}}{6\text{ s}} = \frac{1}{3}$ m/s. Thus the instructions are "Start 2 m from the motion sensor and walk away from the sensor at a steady speed of $\frac{1}{3}$ m/s for 6 s, that is, until you are 4 m from the motion sensor."

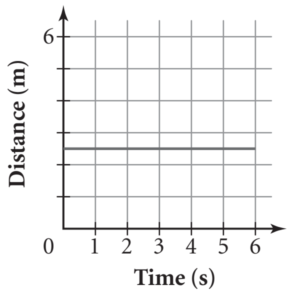
For Graph b, the starting position is 3.5 and the speed is $\frac{0\text{ m} - 3.5\text{ m}}{14\text{ s} - 0\text{ s}} = -\frac{1}{4}$ m/s, so the instructions are "Start 3.5 m from the motion sensor, and walk toward the sensor at $\frac{1}{4}$ m/s for 14 s."

For Graph c, the walking instructions are "Start at the 3 m mark and walk away from the sensor at $\frac{1}{4}$ m/s for 4 s; then walk toward the sensor at 1 m/s for 2 s." Can see you how these directions fit the graph?

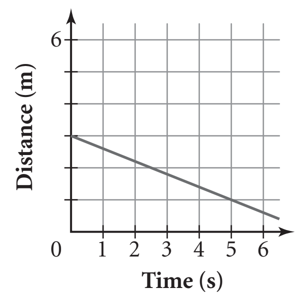
(continued)

Step 2 Here are the graphs for the three sets of walking instructions in Step 2. Be sure you understand how to create these.

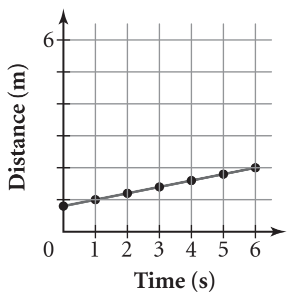
a. The graph starts at (0, 2.5), and remains at the same distance for 6 s.



b. The graph starts at (0, 3). The walker walks toward the sensor at 0.4 m/s for 6 s, so the distance decreases by $0.4 \cdot 6 = 2.4$ m. Thus the ending point is (6, 0.6).



c. Plot and connect the points given.

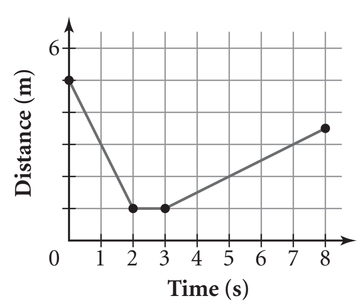


Step 3 The recursive routine for the table in Step 2c has starting value 0.8, and the rule is "add 0.2."

Here is an example of a more complicated walk.

EXAMPLE Graph a walk from the set of instructions "Start at the 5-meter mark. Walk toward the motion sensor at a steady 2 meters per second for 2 seconds. Stand still for 1 second. Then walk away from the sensor at a steady 0.5 meter per second for 5 seconds."

▶ ***Solution*** Think about where the walker starts and how much distance will be covered during each of the three portions of the walk. The graph starts at (0, 5). Then the walker travels 2 m/s for 2 s, so she travels 4 m toward the sensor. Thus there is a straight line connecting (0, 5) to $(0 + 2, 5 - 4) = (2, 1)$. Next, she stands still for 1 s, meaning her distance doesn't change. So connect the previous point to $(2 + 1, 1 + 0) = (3, 1)$. Finally, she walks 0.5 m/s for 5 s, so she travels 2.5 m. Connect the previous point to $(3 + 5, 1 + 2.5) = (8, 3.5)$.



CONDENSED LESSON

3.4 Linear Equations and the Intercept Form

In this lesson you will

- **write linear equations** from recursive routines
- learn about the **intercept form** of a linear equation, $y = a + bx$
- observe how the values of a and b in the intercept form relate to the graph of the equation

Investigation: Working Out with Equations

Manisha burned 215 calories on her way to the gym. At the gym, she burns 3.8 calories per minute by riding a stationary bike.

Steps 1–3 You can use the following calculator routine to find the total number of calories Manisha has burned after each minute she pedals.

Press {0, 215} [ENTER].

Press Ans + {1, 3.8}.

Press [ENTER] repeatedly.

In the list {0, 215}, 0 is the starting *minutes* value and 215 is the starting *calories* value. Ans + {1, 3.8} adds 1 to the minute value and 3.8 to the calorie value each time you press [ENTER].

You can use your calculator routine to generate this table.

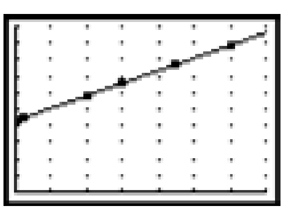
Manisha's Workout

Pedaling time (min), x	Total calories burned, y
0	215
1	218.8
2	222.6
20	291
30	329
45	386
60	443

Steps 4–7 In 20 minutes, Manisha has burned 215 + 3.8(20) or 291 calories. In 38 minutes, she has burned 215 + 3.8(38) or 359.4 calories. Writing and evaluating expressions like these allows you to find the calories burned for any number of minutes without having to find all the previous values.

If x is the time in minutes and y is the number of calories burned, then $y = 215 + 3.8x$. Check that this equation produces the values in the table by substituting each x-value to see if you get the corresponding y-value.

Steps 8–10 Use your calculator to plot the points from your table. Then, enter the equation $y = 215 + 3.8x$ into the Y= menu and graph it. The line should pass through all the points as shown.



[0, 70, 10, 0, 500, 50]

Note that it makes sense to draw a line through the points because Manisha is burning calories every instant she is pedaling.

(continued)

If you substitute 538 for y in the equation, you get $538 = 215 + 3.8x$. You can work backward from 538, undoing each operation, to find the value of x.

$538 = 215 + 3.8x$	Original equation.
$323 = 3.8x$	Subtract 215 to undo the addition.
$85 = x$	Divide by 3.8 to undo the multiplication.

Manisha must pedal 85 minutes to burn 538 calories.

Look back at the recursive routine, the equation, and the graph. The starting value of the recursive routine, 215, is the constant value in the equation and the y-value where the graph crosses the y-axis. The recursive rule, "add 3.8," is the number x is multiplied by in the equation. In the graph, this rule affects the steepness of the line—you move up 3.8 units for every 1 unit you move to the right.

In your book, read the text and examples after the investigation. Make sure you understand the **intercept form** of an equation, $y = a + bx$, and how the ***y*-intercept,** a, and the **coefficient,** b, are reflected in a graph of the equation. Here is an additional example.

EXAMPLE

A plumber charges a fixed fee of $45 for coming to the job, plus $30 for each hour he works.

a. Define variables and write an equation in intercept form to describe the relationship. Explain the real-world meaning of the values of a and b in the equation.

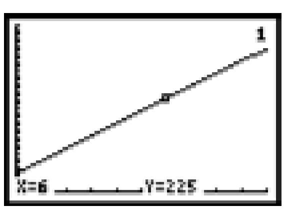
b. Graph your equation. Use your graph to find the number of hours the plumber works for $225.

c. Describe how your equation and graph would be different if the plumber did not charge the $45 fixed fee.

▶ ***Solution***

a. If x represents the hours worked and y represents the total charge, the equation is $y = 45 + 30x$. The value of a, which is 45, is the fixed fee. The value of b, which is 30, is the hourly rate.

b. Here is the graph. To find the number of hours the plumber works for $225, trace the graph to find the point with y-value 225. The corresponding x-value, 6, is the number of hours.



c. If the plumber did not charge a fixed fee, the a-value would be 0 and the equation would be $y = 30x$. The line would have the same steepness, but because the charge for 0 hours would be $0, it would pass through the origin (that is, the y-intercept would be 0).

CONDENSED LESSON

3.5 Linear Equations and Rate of Change

In this lesson you will

- use the **rate of change** to write a linear equation for a situation
- learn how the rate of change relates to a linear equation and graph
- observe how the *a*-value in $y = a + bx$ relates to the graph

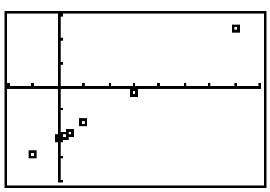
Page 187 of your book shows linear equations in intercept form for some of the situations you have explored in this chapter. For each situation, think about what the variables represent and what the values of a and b mean.

On a cold, windy day, the temperature you feel is colder than the actual temperature because of the wind chill factor. In this lesson you'll look at the relationship between actual temperatures and wind chills. To start, read and follow along with Example A in your book.

Investigation: Wind Chill

Steps 1–4 The table on page 188 of your book relates approximate wind chills to different actual temperatures when the wind speed is 20 miles per hour. Let the input variable, x, be the actual temperature in °F, and let the output variable, y, be the wind chill temperature in °F.

Below is a plot of the data in the viewing window [−10, 40, 5, −40, 30, 10].



To generate the values on your calculator, you can use the following routine:

Press {−5, −28.540} ENTER.
Press {Ans(1) + 1, Ans(2) + 1.312}.
Press ENTER repeatedly.

The starting list, {−5, −28.540}, represents −5°F and its wind chill equivalent. The routine finds wind chill equivalents for temperature of −5°, −4°, −3°, and so on. Each time the actual temperature increases by 1, the wind chill increases by 1.312.

In the table on the next page, columns have been added showing the change in consecutive input and output values and in the rate of change.

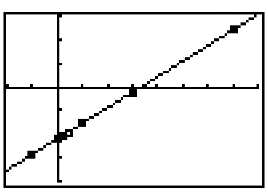
(continued)

Input	Output	Change in input values	Change in output values	Rate of change
−5	−28.540			
0	−21.980	5	6.56	$\frac{6.56}{5} = 1.312$
1	−20.668	1	1.312	$\frac{1.312}{1} = 1.312$
2	−19.356	1	1.312	$\frac{1.312}{1} = 1.312$
5	−15.420	3	3.936	$\frac{3.936}{3} = 1.312$
15	−2.300	10	13.12	$\frac{13.12}{10} = 1.312$
35	23.940	20	26.24	$\frac{26.24}{20} = 1.312$

Steps 5–8 The rate of change is 1.312, meaning that the wind chill temperature increases by 1.312° for each increase of 1° in the actual temperature. The equation that relates the wind chill x to the actual temperature y is $y = -21.980 + 1.312x$.

The equation $y = -21.980 + 1.312x$ is written in intercept form, $y = a + bx$. Notice that the rule for the recursive routine, "add 1.312," appears as the b-value in the equation. The starting value of the routine, −21.980, is *not* the value of a in the equation. The value of a is −21.980, the wind chill when the actual temperature is 0°.

Below, the graph of $y = -21.980 + 1.312x$ has been added to the scatter plot. It makes sense to draw a line through the points because every possible temperature has a wind chill equivalent.



Notice that the y-intercept of the graph, −21.980, is the value of a in the equation.

As you have seen, the rate of change, 1.312, appears as the value of b, or the coefficient of x, in the equation. In the graph, the rate of change is the number of units you move up each time you move 1 unit to the right.

You can use the rate 1.312 to find the actual temperature corresponding to a wind chill of 9.5°. First, note that a wind chill of −2.3° corresponds to an actual temperature of 15°. To get from a wind chill of −2.3° to a wind chill of (approximately) 9.5°, you must add 1.312 nine times, that is, $-2.3 + 9(1.312) = 9.508 \approx 9.5$. Each increase of 1.312° in wind chill corresponds to a 1° increase in actual temperature. So the actual temperature corresponding to a wind chill of 9.5° is approximately $15 + 9(1)$, or 24°.

Example B walks you through a situation similar to the one you looked at in the investigation. Work through this example carefully and make sure you understand it.

CONDENSED LESSON 3.6

Solving Equations Using the Balancing Method

In this lesson you will

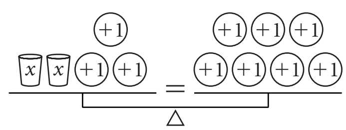
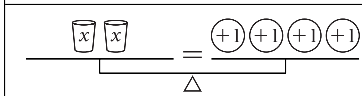
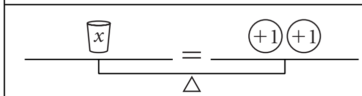
- use a balance scale to **model solving an equation**
- solve equations by using the **balancing method**
- compare several methods for solving the same equation

You have found the solutions of linear equations by tracing graphs, by looking at tables, and by working backward to undo the operations. In this lesson you explore how to solve equations using the **balancing method.**

Investigation: Balancing Pennies

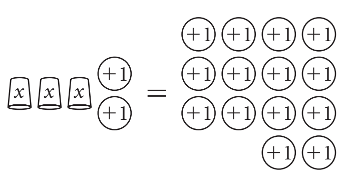
The drawing of a balance on page 195 of your book is a visual model of the equation $2x + 3 = 7$. A cup represents the variable x, and pennies represent the numbers. Each cup contains the same number of pennies. To solve the equation, find the number of pennies in each cup.

Steps 1–3 The pictures below show one way to solve the equation. Note that at each stage, the same thing must be done to both sides so that the scale remains balanced.

Picture	Action taken	Equation
	Original balance.	$2x + 3 = 7$
	Remove 3 pennies from each side.	$2x = 4$
	Remove half of each side.	$x = 2$

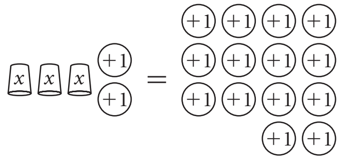
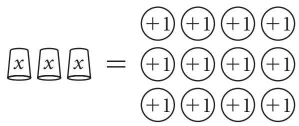
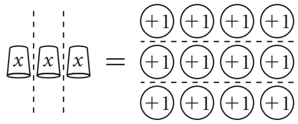
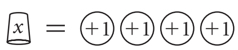
There are 2 pennies in each cup, so 2 is the solution to the original equation.

Steps 4–8 You can create a pennies-and-cups equation. First, draw a large equal sign and put the same number of pennies on each side. On one side, put some of the pennies into three identical stacks, leaving at least a few pennies, and then hide each stack under a paper cup. Here is the arrangement one group made.



This setup models the equation $3x + 2 = 14$. You can solve the equation—that is, find the number of pennies under each cup—by doing the same thing on both sides of the equal sign. (Think of this as a balance scale; you need to do the same thing to both sides so that the scale remains balanced.)

(continued)

Picture	Action taken	Equation
	Original setup.	$3x + 2 = 14$
	Remove 2 pennies from each side.	$3x = 12$
	Divide each side by 3.	$\frac{3x}{3} = \frac{12}{3}$
	Reduce (leaving a third of each side).	$x = 4$

The solution to $3x + 2 = 14$ is 4. Check this by substituting 4 for x.

The model you used in the investigation works only when the numbers involved are whole numbers. In your book, the text following the investigation and Example A show how you can use a similar model to solve equations involving negative integers. Read this material and make sure you understand it.

Once you get used to doing the same thing to both sides of an equation, you can use the balancing method without drawings or models. This allows you to solve equations involving fractions or negative numbers. Example B in your book shows you how to solve an equation using all four of the methods you know so far. Read through that example. The example below uses the balancing method to solve another equation.

EXAMPLE | Solve $7.4 - 20.2x = -1.69$ using the balancing method.

▶ ***Solution***

$7.4 - 20.2x = -1.69$	Original equation.
$-7.4 + 7.4 - 20.2x = -1.69 + -7.4$	Add -7.4 to both sides.
$-20.2x = -9.09$	Combine like terms. (Evaluate and remove the zero.)
$\frac{-20.2x}{-20.2} = \frac{-9.09}{-20.2}$	Divide both sides by -20.2.
$x = 0.45$	Reduce.