CONDENSED LESSON

# 4.1 A Formula for Slope

In this lesson you will

- learn how to calculate the **slope** of a line given two points on the line
- determine whether a point lies on the same line as two given points
- find a point on a line given a known point on the line and the slope

In Chapter 4, you saw that the rate of change of a line can be a numerical and graphical representation of a real-world change like a car's speed. Look at the lines and equation shown on page 215 of your book. Because the coefficient of *x* represents the rate of change of the line, you can match the equations to the lines by looking at the coefficients—the greater the coefficient, the steeper the line.

The rate of change of a line is often referred to as its **slope.** You can find the slope of a line if you know the coordinates of two points on the line.

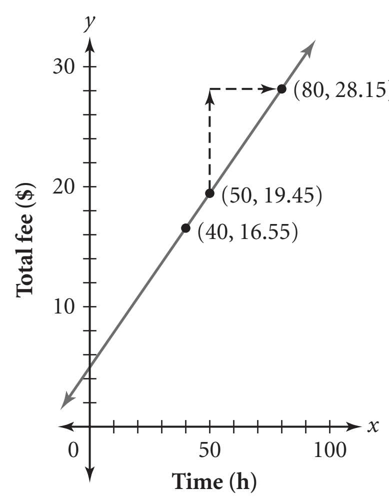
## Investigation: Points and Slope

Hector pays a flat monthly charge plus an hourly rate for Internet service. The table on page 215 of your book shows Hector's total monthly fee for three months. Because the hourly rate is constant, this is a linear relationship. To find the rate in dollars per hour, divide the change in fee by the change in time for two months. For example, using October and November, you get

$$\frac{28.15 - 19.45}{80 - 50} = \frac{8.70}{30} = 0.29$$

So the rate is \$0.29 per hour. Verify that you get this same result if you use the data for September and October or September and November.

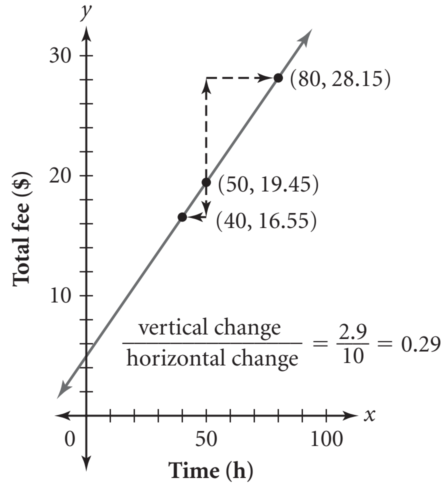
Here is a graph of the Internet data with a line drawn through the points. The arrows show how you can move from (50, 19.45) to (80, 28.15) using one vertical move and one horizontal move.




The length of the vertical arrow is 28.15 − 19.45 or 8.70 units, which is the change in total fees from October to November. The length of the horizontal arrow is 80 − 50 or 30 units, which is the change in the number of hours from October to November. Notice that the lengths are the two quantities we divided to find the hourly rate. This hourly rate, 0.29, is the *slope* of the line. The right triangle created by drawing arrows to show vertical and horizontal change is called a **slope triangle.**

(continued)

At right is the same graph with arrows drawn between the points for September and October. Notice that the slope—or the vertical change divided by the horizontal change—is the same. *Any pair of points on a line will give the same slope.*




You can find the vertical change and the horizontal change by subtracting *corresponding* coordinates. If one of the points is $(x_1, y_1)$ and the other is $(x_2, y_2)$, then the slope is

$$\frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y_1 - y_2}{x_1 - x_2}$$

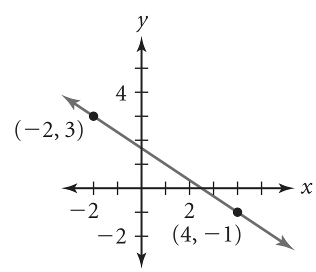
In the investigation, the slope of the line is positive. The example in your book involves a line with negative slope. Read this example carefully. Here is another example.

***EXAMPLE*** Consider the line through $(-2, 3)$ and $(4, -1)$.

**a.** Find the slope of the line.

**b.** Without graphing, verify that $\left(3, -\frac{1}{3}\right)$ is on the line.

**c.** Find the coordinates of another point on the line.




***▸ Solution***

**a.** Slope $= \dfrac{-1 - 3}{4 - (-2)} = \dfrac{-4}{6} = -\dfrac{2}{3}$

**b.** The slope between any two points is the same. So if the slope between $\left(3, -\frac{1}{3}\right)$ and either of the original points is $-\frac{2}{3}$, then the point is on the line. The slope between $\left(3, -\frac{1}{3}\right)$ and $(4, -1)$ is

$$\frac{-1 - \left(-\frac{1}{3}\right)}{4 - 3} = \frac{-\frac{2}{3}}{1} = -\frac{2}{3}$$

So $\left(3, -\frac{1}{3}\right)$ is on the line.

**c.** Start with one of the original points. Add the change in $x$ to the $x$-coordinate and the change in $y$ to the $y$-coordinate. Let's start with $(4, -1)$.

$(4 + \text{change in } x, -1 + \text{change in } y) = (4 + 6, -1 + (-4)) = (10, -5)$

So $(10, -5)$ is on the line.

Read the rest of Lesson 4.1 in your book. Make sure you understand how you can tell by looking at a line whether the slope is positive, negative, zero, or undefined. Note that when the equation for a line is written in the form $y = a + bx$, the letter $b$ represents the slope.

CONDENSED LESSON 4.2

# Writing a Linear Equation to Fit Data

In this lesson you will

- find a **line of fit** for a set of data
- use a **linear model** to make predictions
- learn about the **slope-intercept** form of an equation

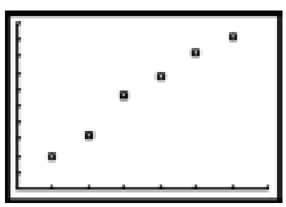
Real-world data rarely fall exactly on a line. However, if the data show a linear pattern, you can find a line to model the data. Such a line is called a **line of fit** for the data. Read the information about lines of fit on page 225 of your book.

## Investigation: Beam Strength

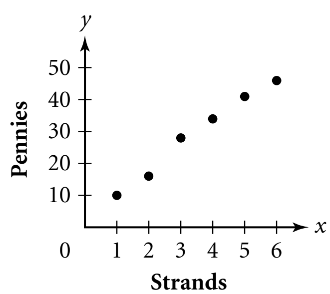
**Steps 1–3** In this investigation, students make "beams" from different numbers of spaghetti strands. Then they test each beam to see how many pennies it can hold before breaking. At right is the data collected by one group of students.

| Number of strands | Number of pennies |
|---|---|
| 1 | 10 |
| 2 | 16 |
| 3 | 28 |
| 4 | 34 |
| 5 | 41 |
| 6 | 46 |

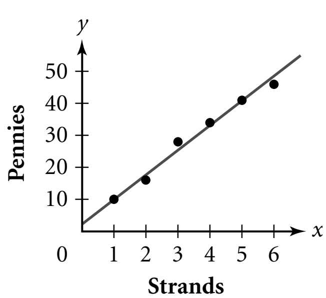
**Steps 4–8** You can make a scatter plot of the data on a graphing calculator or on paper.



[0, 7, 1, 0, 50, 5]

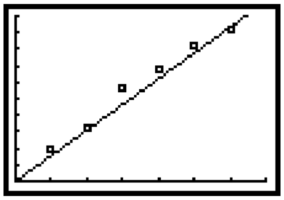



By moving a strand of spaghetti around on your sketch, find a line you think fits the data. Make your line go through two points. Use the coordinates of the two points to find the slope of the line. In the example below, the line goes through (1, 10) and (5, 41). The slope is $\frac{41-10}{5-1}$ or 7.75.

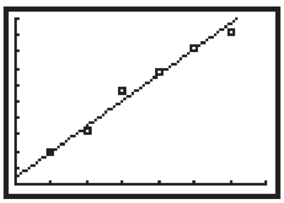



**(continued)**

Use the slope, $b$, to graph the equation $y = bx$ on your calculator. For the preceding line, this is $y = 7.75x$.



Notice that this line is a little too low to fit the data well. Using the sketch on the preceding page, you can estimate that the $y$-intercept, $a$, of the line of fit is about 2. So the equation of the line of fit in intercept form is about $y = 2 + 7.75x$. Graph this equation on your calculator. This line appears to be a good fit. In some cases you'll need to adjust the intercept value several times until you are happy with the fit.



Note that the equation you end up with depends on the two data points your line passes through and your estimate of the $y$-intercept. Different people will find different equations.

**Steps 9–12** The line of fit found above, $y = 2 + 7.75x$, is a *model* for the relationship between the number of strands in the beam, $x$, and the number of pennies the beam supports, $y$. The slope, 7.75, represents the increase in the number of pennies each time you add one strand to the beam.

You can use the line of fit to make predictions. To predict the number of strands needed to support \$5 worth of pennies (500 pennies), trace the graph to find the $x$-value corresponding to a $y$-value of 500, or solve $500 = 7.75x + 2$. It would take about 64 strands to support this much weight.

The model predicts that a 10-strand beam can hold $7.75(10) + 2$ or about 80 pennies. A 17-strand beam can hold $7.75(17) + 2$ or about 134 pennies.

Now, follow along with the example in your book. It shows you how to fit a line to a different data set.

To find the line of fit in the investigation, we started with the slope and then found the $y$-intercept. Because of the importance of slope, many people use the **slope-intercept form** of a linear equation, which shows the slope before the $y$-intercept. The slope-intercept form is $y = mx + b$. In this form, $m$ represents the slope and $b$ represents the $y$-intercept.

CONDENSED LESSON

# 4.3 Point-Slope Form of a Linear Equation

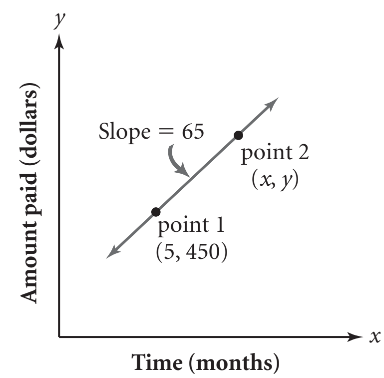
In this lesson you will

- write equations in **point-slope form**
- find the equation of a line given one point on the line and the slope
- find the equation of a line given two points on the line

If you are given the slope and the $y$-intercept of a line, it is easy to write an equation for the line. The example in your book shows you how to find an equation when you know one point and the slope. Here is an additional example.

***EXAMPLE*** When Rosi bought her computer, she made a down payment and then made payments of \$65 per month. After 5 months, she had paid \$450. After 18 months, her computer was paid for. What is the total amount Rosi paid for her computer?

***Solution*** Because the rate of change is constant (\$65 per month), you can model this relationship with a linear equation. Let $x$ represent time in months, and let $y$ represent the amount paid.




The problem gives the slope, 65, and one point, (5, 450). Let $(x, y)$ represent a second point on the line, and use the slope formula, $\frac{y_2 - y_1}{x_2 - x_1} = b$, to find a linear equation.

$\frac{y - 450}{x - 5} = 65$ — Substitute the coordinates of the points.

$y - 450 = 65(x - 5)$ — Multiply by $(x - 5)$ to undo the division.

$y = 450 + 65(x - 5)$ — Add 450 to undo the subtraction.

The equation $y = 450 + 65(x - 5)$ gives the total amount paid, $y$, in $x$ months. To find the total amount Rosi paid in 18 months, substitute 18 for $x$.

$$y = 450 + 65(x - 5) = 450 + 65(18 - 5) = 450 + 65(13) = 450 + 845 = 1295$$

Rosi paid a total of \$1295 for her computer.

The equation $y = 450 + 65(x - 5)$ is in **point-slope form.** Read about point-slope form on page 235 of your book.

**(continued)**

## Investigation: The Point-Slope Form for Linear Equations

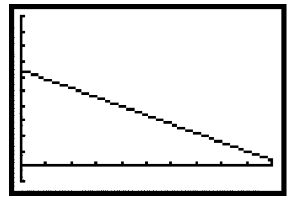
**Steps 1–5** Jenny is moving at a constant rate. The table on page 235 of your book shows her distances from a fixed point after 3 seconds and after 6 seconds.

The slope of the line that represents this situation is $\frac{2.8-4.6}{6-3}$ or $-0.6$.

If you use the point (3, 4.6), the equation for this situation in point-slope form is $y = 4.6 - 0.6(x - 3)$.

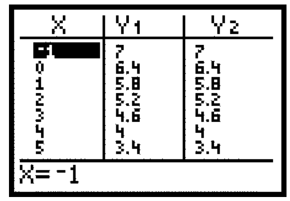
If you use (6, 2.8), the equation is $y = 2.8 - 0.6(x - 6)$.

Enter both equations into your calculator and graph them. You will see only one line, which indicates that the equations are equivalent.

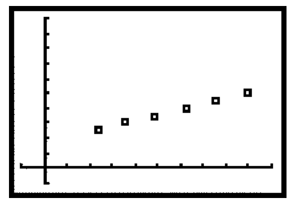


[0, 10, 1, −1, −10, 1]

Now, look at the table for the two equations. Notice that for every $x$-value, the Y1- and Y2-values are the same. This also indicates that the equations $y = 4.6 - 0.6(x - 3)$ and $y = 2.8 - 0.6(x - 6)$ are equivalent.




**Steps 6–9** The table on page 236 of your book shows how the temperature of a pot of water changed over time as it was heated. If you plot the data on your calculator, you'll see an approximately linear pattern.

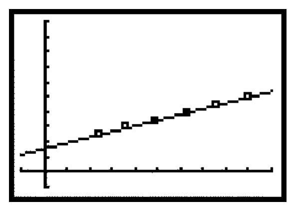


[−10, 100, 10, −10, 100, 10]

Choose a pair of data points. For this example, we'll use (49, 35) and (62, 40). The slope of the line through these points is

$$\frac{40-35}{62-49} = \frac{5}{13} \approx 0.38$$

Using (49, 35), the equation for the line in point-slope form is $y = 35 + 0.38(x - 49)$. If you graph this equation, you'll see that the line fits the data very closely.



Now, pick a different pair of data points. Find an equation for the line through the points, and graph the equation on your calculator. Does one of the equations fit the data better than the other?

CONDENSED LESSON

# 4.4 Equivalent Algebraic Equations

In this lesson you will

- learn about the **distributive property**
- determine whether equations are **equivalent**
- rewrite linear equations in **intercept form**
- use **mathematical properties** to rewrite and solve equations

You have seen that the same line can be represented by more than one equation. In this lesson you'll learn how to recognize equivalent equations by using mathematical properties and the rules for order of operations.

Read the information about the distributive property on page 241 of your book.

## Investigation: Equivalent Equations

Page 241 of your book gives six equations, labeled a–f. Although these equations look very different, they are all equivalent. To see this, use the distributive property to rewrite each equation in intercept form.

**a.** $y = 3 - 2(x - 1)$
$= 3 - 2x + 2$
$= 5 - 2x$

**b.** $y = -5 - 2(x - 5)$
$= -5 - 2x + 10$
$= 5 - 2x$

**c.** $y = 9 - 2(x + 2)$
$= 9 - 2x - 4$
$= 5 - 2x$

**d.** $y = 0 - 2(x - 2.5)$
$= 0 - 2x + 5$
$= 5 - 2x$

**e.** $y = 7 - 2(x + 1)$
$= 7 - 2x - 2$
$= 5 - 2x$

**f.** $y = -9 - 2(x - 7)$
$= -9 - 2x + 14$
$= 5 - 2x$

All the equations are equivalent to $y = 5 - 2x$. To check that each original equation is equivalent to $y = 5 - 2x$, enter both equations into your calculator and graph them. You will get the same line, which indicates that the same values satisfy both equations.

Now, look at equations a–o on page 242 of your book. These equations represent only four different lines. To find the equivalent equations, write each equation in intercept form. You should get these results.

Equations a, i, l, and m are equivalent to $y = -5 + 2x$.
Equations b, d, g, and k are equivalent to $y = 2 + 2x$.
Equations e, j, and n are equivalent to $y = -3 - 6x$.
Equations c, f, h, and o are equivalent to $y = 4 - 6x$.

**(continued)**

In equations h and j, $x$ and $y$ are on the same side of the equation and the other side is a constant. These equations are in **standard form.** Here are the steps for rewriting equation j in intercept form.

| | |
|---|---|
| $12x + 2y = -6$ | Original equation. |
| $2y = -6 - 12x$ | Subtraction property (subtract $12x$ from both sides). |
| $y = \frac{-6 - 12x}{2}$ | Division property (divide both sides by 2). |
| $y = -3 - 6x$ | Distributive property (divide each term by 2). |

In the investigation, you saw that no matter what form a linear equation is given in, you can rewrite it in intercept form. When equations are in intercept form, it is easy to see whether they are equivalent. Page 243 of your book reviews the properties that allow you to rewrite (and solve) equations. Read these properties and the examples that follow. Here are two more examples.

***EXAMPLE C*** Is $21x + 3y = 12$ equivalent to $y = -10 - 7(x - 1)$?

▶ ***Solution***

Rewrite $21x + 3y = 12$ in intercept form.

| | |
|---|---|
| $21x + 3y = 12$ | Original equation. |
| $3y = 12 - 21x$ | Subtraction property (subtract $21x$ from both sides). |
| $y = \frac{12 - 21x}{3}$ | Division property (divide both sides by 3). |
| $y = 4 - 7x$ | Distributive property (divide 12 by 3 and $-21x$ by 3). |

Rewrite $y = -10 - 7(x - 1)$ in intercept form.

| | |
|---|---|
| $y = -10 - 7(x - 1)$ | Original equation. |
| $y = -10 - 7x + 7$ | Distributive property. |
| $y = -3 - 7x$ | Add $-10$ and 7. |

The equations are not equivalent.

***EXAMPLE D*** Solve $\frac{4(2x - 3)}{7} = 4$. Identify the property of equality used in each step.

▶ ***Solution***

| | |
|---|---|
| $\frac{4(2x - 3)}{7} = 4$ | Original equation. |
| $4(2x - 3) = 28$ | Multiplication property (multiply both sides by 7). |
| $2x - 3 = 7$ | Division property (divide both sides by 4). |
| $2x = 10$ | Addition property (add 3 to both sides). |
| $x = 5$ | Division property (divide both sides by 2). |

CONDENSED LESSON 4.5

# Writing Point-Slope Equations to Fit Data

In this lesson you will

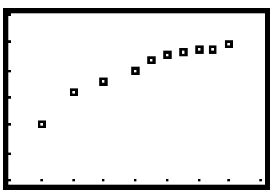
- write **point-slope equations** to fit data
- use equations of lines of fit to make predictions
- compare two methods for fitting lines to data

This lesson gives you more practice using the point-slope form to model data. You may find that using the point-slope form is more efficient than using the intercept form because you don't have to first write a direct variation equation and then adjust it for the intercept.

## Investigation: Life Expectancy

**Steps 1–4** The table on page 248 of your book shows the relationship between the number of years a person might be expected to live and the year he or she was born.

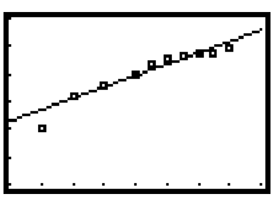
Plot the female life-expectancy data in the window [1930, 2010, 10, 55, 85, 5].



Look for two points on the graph so that the line through the points closely reflects the pattern of all the points. For this example, we'll use (1970, 74.7) and (1990, 78.8). The slope of the line through these points is

$$\frac{78.8 - 74.7}{1990 - 1970} = \frac{4.1}{20} = 0.205$$

Using (1970, 74.7), you can write the equation $y = 74.7 + 0.205(x - 1970)$.



To predict the life expectancy of a female who will be born in 2022, substitute 2022 for $x$ in the equation.

$$\begin{aligned} y &= 74.7 + 0.205(x - 1970) \\ &= 74.7 + 0.205(2022 - 1970) \\ &= 74.7 + 0.205(52) \\ &= 74.7 + 10.66 = 85.36 \end{aligned}$$

The equation predicts that a female born in 2022 will have a life expectancy of 85.36 years.

(continued)

**Steps 5–8** If we had chosen a different pair of points, we would have found a different equation and made a different life-expectancy prediction. For example, if we had used (1950, 71.1) and (1995, 78.9), we would have gotten the slope $\frac{78.9 - 71.1}{1995 - 1950} = \frac{7.8}{45} \approx 0.173$ and the equation $y = 71.1 + 0.173(x - 1950)$. This equation gives a life-expectancy prediction of about 83.56 years for a female born in 2022.

Now, find equations of lines of fit for the male data and combined data. Here are equations for all three sets of data, using the points for 1970 and 1990.

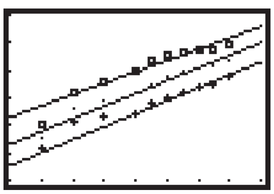
Female: $y = 74.7 + 0.205(x - 1970)$

Male: $y = 67.1 + 0.235(x - 1970)$

Combined: $y = 70.8 + 0.23(x - 1970)$

Notice that the slopes of all three lines are close to 0.2, indicating that life expectancy increases by about 0.2 year for each age increase of 1 year, regardless of gender.

The graph below shows all three sets of data and all three lines graphed in the same window. The data for females is plotted with boxes, the data for males is plotted with plus signs, and the combined data is plotted with dots.



Notice that the line for the combined data is between the other two lines. This is reasonable because the combined life expectancy for both males and females should be between the life expectancy for females and the life expectancy for males.

You have used two methods for finding the equation of a line of fit. One method uses the intercept form, and the other uses the point-slope form. For the intercept-form method (which you used in the Investigation Beam Strength), you found a line parallel to the line of fit and then adjusted it up or down, using estimation, to fit the points. With the point-slope method, you get an equation without making any adjustments, but you may find the line does not fit the data as well as you would like.

CONDENSED LESSON

# 4.6 More on Modeling

In this lesson you will

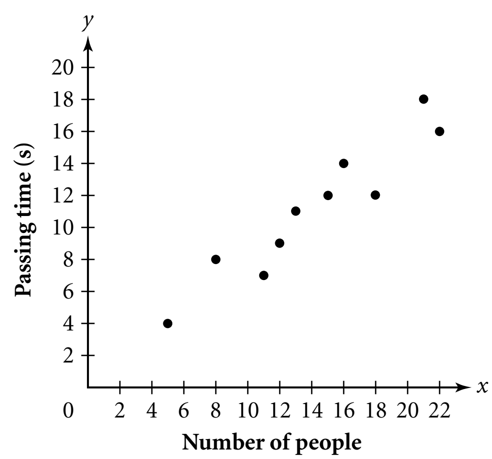
- use **Q-points** to fit a line to a set of data
- use a line of fit to make predictions

Several times in this chapter you have found the equation of a line of fit for a data set. You probably found that you and your classmates often wrote different equations even though you were working with the same data. In this investigation you will learn a systematic method for finding the equation of a line of fit. This method always gives the same equation for a given set of data.

## Investigation: Bucket Brigade

**Steps 1–3** In this investigation, students form a line and record the time it takes to pass a bucket from one end of the line to the other. After each trial, some students sit down and the remaining students repeat the experiment. Here are the data one class collected and the graph of the data.

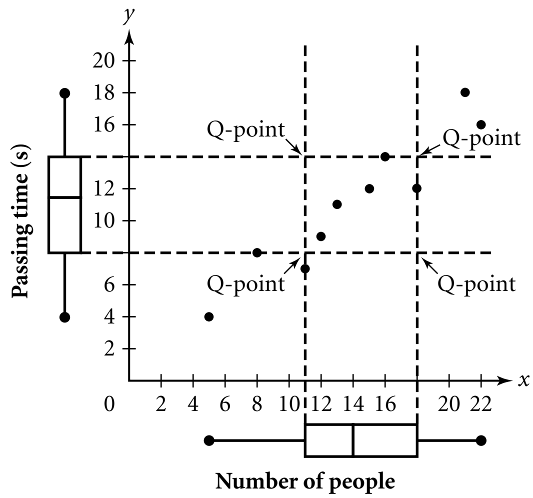
| Number of people, $x$ | Passing time (s), $y$ |
|---|---|
| 22 | 16 |
| 21 | 18 |
| 18 | 12 |
| 16 | 14 |
| 15 | 12 |
| 13 | 11 |
| 12 | 9 |
| 11 | 7 |
| 8 | 8 |
| 5 | 4 |




**Steps 4–13** You can use the quartiles of the $x$- and $y$-values to fit a line to the data. First, find the five-number summaries. The five-number summary for the $x$-values is 5, 11, 14, 18, 22. The five-number summary for the $y$-values is 4, 8, 11.5, 14, 18. Use these values to add horizontal and vertical box plots to the graph.

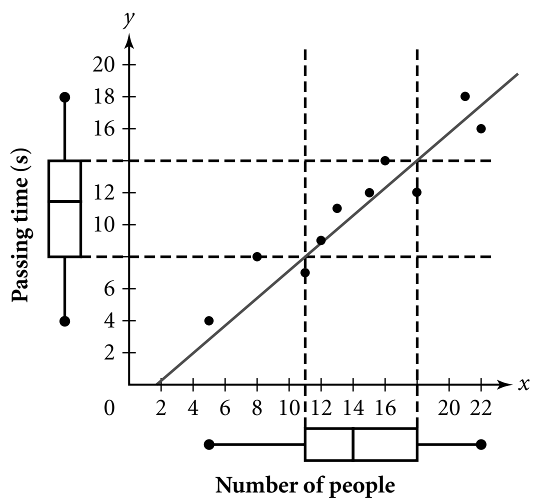
**(continued)**

Next, draw vertical lines from the Q1- and Q3-values on the $x$-axis box plot, and draw horizontal lines from the Q1- and Q3-values on the $y$-axis box plot. These lines form a rectangle. The vertices of this rectangle are called **Q-points.** Q-points may or may not be actual points in the data set. Note that anyone who starts with this data will get the same Q-points.




Finally, draw the diagonal of the rectangle that reflects the direction of the data. This is a line of fit for the data.

Use the coordinates of the Q-points (11, 8) and (18, 14) to write an equation for this line. The slope is $\frac{14-8}{18-11}$ or $\frac{6}{7}$, so one equation is $y = 8 + \frac{6}{7}(x - 11)$. In intercept form, this is $y = -1\frac{3}{7} + \frac{6}{7}x$. The slope represents the number of seconds it takes each person to pass the bucket. The $y$-intercept represents the amount of time it takes 0 people to pass the bucket. In this case, the $y$-intercept is a negative number, which doesn't make sense in the real situation. This demonstrates that a model only *approximates* what actually happens.




Now, use your calculator to plot the data points, draw vertical and horizontal lines through the quartile values, and find a line of fit. (See **Calculator Note 4B** for help using the draw menu.)

The example in your book uses the Q-point method to fit a line to a different set of data. Read this example and make sure you understand it.

CONDENSED LESSON

# 4.7 Applications of Modeling

In this lesson you will

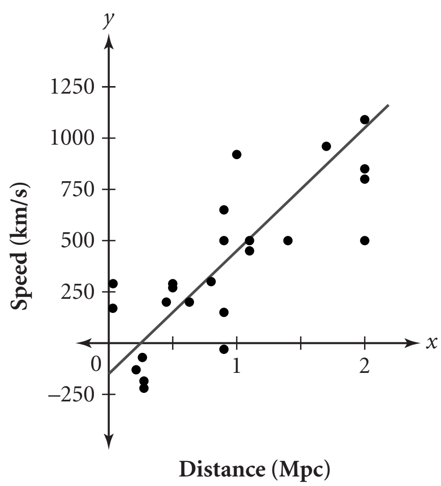
- use and compare three methods of finding a line of fit for a set of data
- use linear models to make predictions

You now know several methods for fitting a line to data. In this lesson you will practice and compare these methods.

## Investigation: What's My Line?

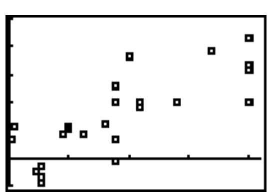
The table on page 261 of your book shows the relationship between a nebula's distance from Earth and the speed at which it is moving away from or toward Earth.

**Steps 1–2** First, you'll find a line of fit by "eyeballing" the data. Plot the data on graph paper. Then, lay a strand of spaghetti on the plot so that it crosses the $y$-axis and follows the direction of the data. Here is an example.




This line crosses the $y$-axis at about $(0, -150)$. The point $(1, 450)$ is also on the line. The slope of the line through these points is $\frac{450 - (-150)}{1 - 0}$ or about 600. So the equation of this line in intercept form is $y = -150 + 600x$.

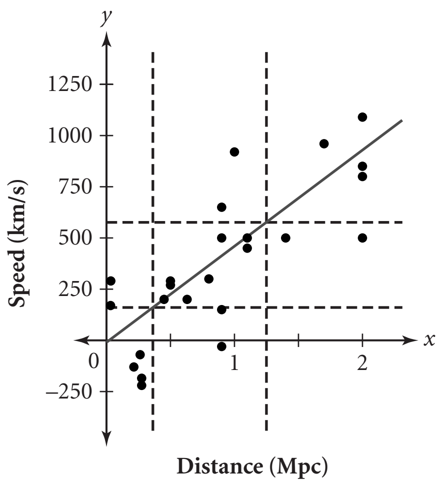
**Steps 3–4** Now, you'll fit a line using "representative" data points. Use your calculator with the window $[0, 2.1, 0.5, -150, 1250, 250]$ to make a scatter plot of the data.



Choose two points you think reflect the direction of the data. For example, you might use $(0.263, -70)$ and $(2, 1090)$. The slope of the line through these points is $\frac{1090 - (-70)}{2 - 0.263}$ or about 667. An equation for the line through these points in point-slope form is $y = -70 + 667(x - 0.263)$. Rewriting this in intercept form gives $y = -245.421 + 667x$.

(continued)

**Steps 5–6** Next, you'll find a line of fit using Q-points. The five-number summary of the $x$-values is 0.032, 0.3625, 0.9, 1.25, 2.0, so Q1 and Q3 are 0.3625 and 1.25. The five-number summary of the $y$-values is $-220$, 160, 295, 575, 1090, so Q1 and Q3 are 160 and 575. Use these values to draw a rectangle on the plot. Then, draw the diagonal that reflects the direction of the data. The diagonal passes through (0.3625, 160) and (1.25, 575).




The slope of this line is $\frac{575 - 160}{1.25 - 0.3625}$ or 468. An equation for this line in point-slope form is $y = 160 + 468(x - 0.3625)$. Rewriting this in intercept form gives $y = -9.65 + 468x$.

**Steps 7–13** In the preceding linear models, the slope indicates the change in speed for each distance increase of 1 megaparsec. The three methods gave different values for the slope (600, 667, and 468). The $y$-intercept of each model represents the speed for a distance of 0. These values are also somewhat different for the three models ($-150$, $-245.421$, and $-9.65$.)

You can use any of the models to predict the distance at which a nebula's speed is 750 km/s. For example, to predict speed using the first model, $y = -150 + 600x$, solve $750 = -150 + 600x$. The solution is 1.5, so a nebula's distance is about 1.5 Mpc if its speed is 750 km/s.

Changing the $y$-intercept of an equation increases or decreases every $y$-coordinate by the same amount. For example, changing the $y$-intercept of the first model, $y = -150 + 600x$, to $-200$ gives the equation $y = -200 + 600x$. The $y$-coordinate of each point on this second line is 50 Mpc less than the $y$-coordinate of that point on the original line. For a distance of 1 Mpc, the first equation gives a speed of 450 km/s and the second equation gives a speed of 400 km/s.

A small change in slope has little effect on points near the data points, but the effect is magnified for points far out on the line, away from the points. You might try changing the slope value of one of the models slightly and then substituting distance values of 3, 4, and so on to see the effect of the change.