6.1 Recursive Routines

In this lesson you will

- explore patterns involving **repeated multiplication**
- write **recursive routines** for situations involving repeated multiplication
- look at **tables and graphs** for situations involving repeated multiplication

In previous chapters you looked at patterns involving repeated addition or subtraction. Such patterns can be modeled with linear equations and straight-line graphs. In this lesson you will begin to explore a different type of pattern.

Investigation: Bugs, Bugs, Everywhere Bugs

Imagine that a bug population starts with 16 bugs and grows by 50% each week. In this investigation you'll look at the pattern of change for this population. In your book, read and follow all the steps of the investigation. Then, look back here to check your results.

This table shows the results for the first 4 weeks.

Bug Invasion

Weeks elapsed	Total number of bugs	Increase in number of bugs (rate of change)	Ratio of this week's total to last week's total
Start (0)	16		
1	24	8	$\frac{24}{16} = \frac{3}{2} = 1.5$
2	36	12	$\frac{36}{24} = \frac{3}{2} = 1.5$
3	54	18	$\frac{54}{36} = \frac{3}{2} = 1.5$
4	81	27	$\frac{81}{54} = \frac{3}{2} = 1.5$

The rate of change for the number of bugs is not constant—it changes from 8 to 12 to 18 to 27—so this pattern is not linear.

Here is a graph of the data. The points have been connected with line segments. Notice that as you move from left to right, the slopes of the line segments increase.

The last column of the table shows that the ratio of the number of bugs each week to the number of bugs the previous week is constant. This constant ratio, 1.5, is the number each week's population is multiplied by to get the next week's population. So, unlike linear patterns, in which you find each value by *adding* a constant number to the previous value, you find each value in this pattern by *multiplying* the previous value by a constant number.

(continued)

Discovering Algebra Condensed Lessons
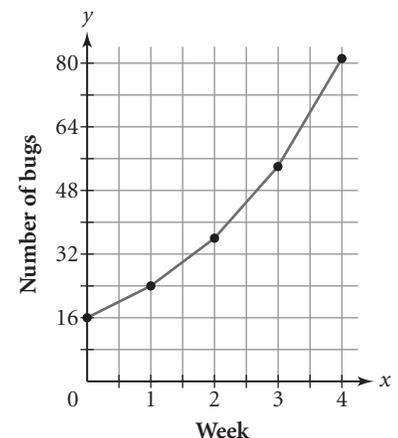
You can model the growth in the bug population with this routine.

Press {0, 16} ENTER

Press {Ans(1) + 1, Ans(2) * 1.5}

Press ENTER to generate each successive term

In the routine, {0, 16} sets the starting bug population (the population for week 0) at 16. The rule {Ans(1) + 1, Ans(2) * 1.5} adds 1 to the week number and multiplies the population by 1.5.

By repeatedly pressing ENTER, you should find that the populations for weeks 5 through 8 are 122, 182, 273, and 410. The populations for weeks 20 and 30 are 53,204 and 4,602,025.

The example in your book involves compound interest, which shows a pattern of increase involving repeated multiplication. Read the example carefully. Then, read the example below, which involves a decreasing pattern.

EXAMPLE | Desi bought a car for $16,000. He estimates that the value of his car will decrease by 15% a year. What will the car be worth after 4 years? After 7 years?

▶ **Solution** | Each year the car's value decreases by 15% of its previous value.

Year	Beginning value		Decrease in value		New value
1	16,000	−	16,000 · 0.15	=	16,000(1 − 0.15) or 13,600
2	13,600	−	13,600 · 0.15	=	13,600(1 − 0.15) or 11,560
3	11,560	−	11,560 · 0.15	=	11,560(1 − 0.15) or 9,826

Each year the car's value is multiplied by 1 − 0.15, or 0.85. In other words, the value at the end of each year is 85% of the previous value. You can model this situation with a recursive routine.

Press {0, 16000} ENTER

Press {Ans(1) + 1, Ans(2) * 0.85}

Press ENTER to generate each successive term

```
{1  13600}
{2  11560}
  {3  9826}
{4  8352.1}
{5  7099.285}
{6  6034.39225}
{7  5129.233413}
■
```

Notice that the value decreases by a smaller amount each year. The value after 4 years is about $8352. The value after 7 years is about $5129.

Exponential Equations

In this lesson you will

- write **exponential equations** to represent situations involving a constant multiplier
- change expressions from **expanded form** to **exponential form**
- use exponential equations to model **exponential growth**

You have used recursive routines to generate patterns involving a constant multiplier. In this lesson you'll learn to represent such patterns with equations. This will allow you to find the value of any term without having to find all the terms before it.

Investigation: Growth of the Koch Curve

Here are Stages 0 through 3 of the Koch curve.
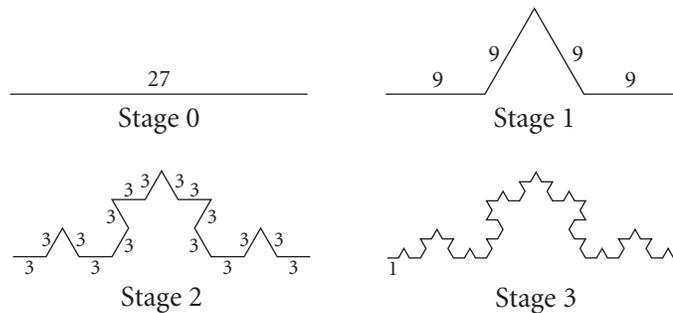
Each stage has four times as many segments as the previous stage, and each segment is $\frac{1}{3}$ the length of the previous segment. This table shows the total length at each stage.

The ratio of the length at each stage to the length at the previous stage is $\frac{4}{3}$. So the length at each stage is $\frac{4}{3}$ times the length at the previous stage. You can use this fact to find the lengths at Stages 4 and 5.

Stage	Total length (units)	Ratio of this stage's length to previous stage's length
0	27	
1	36	$\frac{36}{27} = \frac{4}{3} = 1.\bar{3}$
2	48	$\frac{48}{36} = \frac{4}{3} = 1.\bar{3}$
3	64	$\frac{64}{48} = \frac{4}{3} = 1.\bar{3}$

$$\text{Stage 4 length} = 64 \cdot \frac{4}{3} = 85.\bar{3}$$

$$\text{Stage 5 length} = 85.\bar{3} \cdot \frac{4}{3} = 113.\bar{7}$$

Notice that at Stage 1 you multiply 27, the original length, by $\frac{4}{3}$ once. At Stage 2, you multiply 27 by $\frac{4}{3}$ twice. At Stage 3, you multiply 27 by $\frac{4}{3}$ three times. You can express this pattern using exponents.

$$\text{Stage 1 length} = 27 \cdot \frac{4}{3} = 27 \cdot \left(\frac{4}{3}\right)^1 = 36$$

$$\text{Stage 2 length} = 27 \cdot \frac{4}{3} \cdot \frac{4}{3} = 27 \cdot \left(\frac{4}{3}\right)^2 = 48$$

$$\text{Stage 3 length} = 27 \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} = 27 \cdot \left(\frac{4}{3}\right)^3 = 64$$

(continued)

In each case the stage number is equal to the exponent. So the length at Stage 5 is $27 \cdot \left(\frac{4}{3}\right)^5 = 113.\overline{7}$, which agrees with the answer on the previous page.

If x is the stage number and y is the total length, then the equation $y = 27 \cdot \left(\frac{4}{3}\right)^x$ models the length at any stage. Here are a calculator graph and a table for this equation.
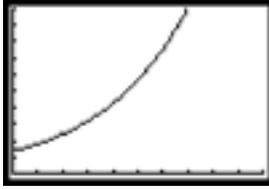
In your book, read the text and examples that follow the investigation. Example A shows how to change expressions from **expanded form** to **exponential form.** Example B explores a situation involving **exponential growth.** Make sure you understand the equation for exponential growth given in the box on page 344. Here is another example.

EXAMPLE   Six years ago, Dawn's grandfather gave her a coin collection worth $350. Since then, the value of the collection has increased by 7% per year. How much is the collection worth now?

▶ **Solution**   You can model this situation with this equation.

Final value

Number of years

$$y = 350(1 + 0.07)^x$$

Starting value

Rate of growth

To find the current value of the collection—that is, the value 6 years after Dawn received it—substitute 6 for x.

$y = 350(1 + 0.07)^x$ Original equation.

$y = 350(1 + 0.07)^6$ Substitute 6 for x.

$y = 350 \cdot 1.07^6$ Add inside the parentheses.

$y \approx 525.26$ Evaluate the expression.

The collection is now worth $525.26.

Multiplication and Exponents

In this lesson you will

- use the **multiplication property of exponents** to rewrite expressions
- use the **power properties of exponents** to rewrite expressions

Suppose a savings account starts with a balance of $500 and earns 3% interest a year. If no money is deposited or withdrawn, the balance after 4 years is $500(1 + 0.03)^4$. Here are two ways you could represent the balance after 5 years.

- You can write $500(1 + 0.03)^5$.
- You can think recursively: The balance after 5 years is the balance after 4 years times the constant multiplier $(1 + 0.03)$. This gives $500(1 + 0.03)^4 \cdot (1 + 0.03)$.

This means that $500(1 + 0.03)^4 \cdot (1 + 0.03) = 500(1 + 0.03)^5$.

In general, you can advance exponential growth by one time period either by multiplying the previous amount by the base or by increasing the exponent by 1. In the investigation you will extend this idea by exploring what happens when you advance by more than one time period.

Investigation: Moving Ahead

Steps 1–3 Look at the expressions in Step 1 in your book. You can write each expression in exponential form with a single base. To see how, first rewrite each expression in expanded form.

a. $3^4 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3)(3 \cdot 3) = 3^6$

b. $x^3 \cdot x^5 = (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x \cdot x) = x^8$

c. $(1 + 0.05)^2 \cdot (1 + 0.05)^4 =$
$[(1 + 0.05) \cdot (1 + 0.05)][(1 + 0.05) \cdot (1 + 0.05) \cdot (1 + 0.05) \cdot (1 + 0.05)] =$
$(1 + 0.05)^6$

d. $10^3 \cdot 10^6 = (10 \cdot 10 \cdot 10) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 10^9$

In each case, you add the exponents in the original expression to get the exponent in the final expression. You can generalize these findings as

$b^m \cdot b^n = b^{m+n}$

Steps 4–5 Now complete parts a–c of Step 4 in your book. Here are the answers.

a. If $16(1 + 0.5)^5$ is the number of bugs in the colony after 5 weeks, then $16(1 + 0.5)^5 \cdot (1 + 0.5)^3$ is the number after 3 more weeks (that is, after a total of 8 weeks). This expression can be rewritten as $16(1 + 0.5)^8$.

b. If $11{,}500(1 - 0.2)^7$ is the value of the truck after 7 years, then $11{,}500(1 - 0.2)^7 \cdot (1 - 0.2)^2$ is its value after 2 more years (that is, after a total of 9 years). This expression can be rewritten as $11{,}500(1 - 0.2)^9$.

c. If $A(1 + r)^n$ represents n time periods of exponential growth, then $A(1 + r)^{n+m}$ models m more time periods (that is, a total of $n + m$ time periods).

Lesson 6.3 • Multiplication and Exponents (continued)

In general, when you are using an exponential model, you can model looking ahead by m time periods by multiplying by $(1 + r)^m$ or by adding m to the exponent.

Page 351 of your book summarizes what you learned in the investigation as the **multiplication property of exponents.** This property is useful for simplifying expressions involving exponents, but keep in mind that it can be used only when the bases are the same. Example A in your book can help you understand why.

Example B illustrates the **power properties of exponents.** Read this example and the following text carefully. Then, read the example below.

EXAMPLE | Use the properties of exponents to rewrite each expression.

a. $(6^4)^3$ b. $(3y)^2$ c. $5^x \cdot 5^y$

d. $7^3 \cdot 5^2 \cdot 7^1$ e. $r^4 s^4$ f. $(p^2)^5$

▶ **Solution** | a. $(6^4)^3 = 6^{4 \cdot 3} = 6^{12}$

b. $(3y)^2 = 3^2 y^2$

c. $5^x \cdot 5^y = 5^{x+y}$

d. $7^3 \cdot 5^2 \cdot 7 = 7^{3+1} \cdot 5^2 = 7^4 \cdot 5^2$

e. $r^4 s^4 = (rs)^4$

f. $(p^2)^5 = p^{2 \cdot 5} = p^{10}$

Scientific Notation for Large Numbers

In this lesson you will

- write large numbers in **scientific notation**
- convert between scientific notation and standard notation
- use scientific notation to simplify calculations with large numbers

The distance from the Sun to the Andromeda galaxy is 13,000,000,000,000,000,000 miles. In this lesson you will learn about **scientific notation,** a method for writing very large numbers like this in a more compact form. In scientific notation, the distance from the Sun to the Andromeda galaxy is written as 1.3×10^{19} miles.

Investigation: A Scientific Quandary

Steps 1–2 Page 355 of your book gives two lists of numbers. The numbers in the first list are in scientific notation. The numbers in the second list are not. Compare the lists to see if you can figure out what it means for a number to be in scientific notation.

Each of the numbers in scientific notation is written as a product of a number between 1 and 10 and a power of 10. Use this idea to determine which of the numbers in Step 1 are in scientific notation.

a. 4.7×10^3 is in scientific notation.

b. 32×10^5 is *not* in scientific notation because 32 is greater than 10.

c. $2^4 \times 10^6$ is *not* in scientific notation because 2^4 is greater than 10 (and because it is written with an exponent).

d. 1.107×10^{13} is in scientific notation.

e. 0.28×10^{13} is *not* in scientific notation because 0.28 is less than 1.

Steps 3–8 Set your calculator to scientific notation mode. (See **Calculator Note 6C.**) When you enter a number and press ENTER, your calculator will convert the number to scientific notation. Use your calculator to convert 5000 and each of the numbers in Step 5 to scientific notation. Here are the results.

Standard notation	Scientific notation
5,000	5×10^3
250	2.5×10^2
$-5,530$	-5.53×10^3
14,000	1.4×10^4
7,000,000	7×10^6
18	1.8×10^1
$-470,000$	-4.7×10^5

(continued)

Notice the following:

- The exponent is equal to the number of digits *after* the first digit in the original number.
- The number multiplied by the power of 10 includes the significant digits of the original number and has one digit to the left of the decimal point.
- If the original number is negative, the number multiplied by the power of 10 is negative.

The exponent 4 is the number of digits after the first digit.
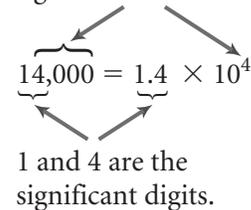
$$14{,}000 = 1.4 \times 10^4$$

1 and 4 are the significant digits.

To convert 415,000,000 to scientific notation, write 4.15 (the significant digits with one digit to the left of the decimal point). Then, count the number of places you have to move the decimal point to get from 415,000,000 to 4.15. Use the result, 8, as the power of 10. So $415{,}000{,}000 = 4.15 \times 10^8$.

To convert 6.4×10^5 to standard notation, move the decimal point five places to the right, adding the zeros you need. So $6.4 \times 10^5 = 640{,}000$.

Read the text and example that follow the investigation in your book. Here is another example.

EXAMPLE   Hemoglobin is a protein in red blood cells that transports oxygen from the lungs to the tissues. There are about 25 trillion red blood cells in the average adult human body, and each red blood cell contains 280 million molecules of hemoglobin. How many molecules of hemoglobin does the average adult human body contain?

▶ **Solution**   First, write the numbers in scientific notation.

$$25 \text{ trillion} = 25{,}000{,}000{,}000{,}000 = 2.5 \times 10^{13}$$
$$280 \text{ million} = 280{,}000{,}000 = 2.8 \times 10^8$$

Now, multiply the numbers.

$$(2.5 \times 10^{13})(2.8 \times 10^8) = 2.5 \times 2.8 \times 10^{13} \times 10^8 \qquad \text{Regroup the numbers.}$$
$$= 7.0 \times 10^{13} \times 10^8 \qquad \text{Multiply 2.5 and 2.8.}$$
$$= 7.0 \times 10^{21} \qquad \text{Use the multiplication property of exponents.}$$

There are about 7.0×10^{21} molecules of hemoglobin in the human body.

6.5 Looking Back with Exponents

In this lesson you will

- use the **division property of exponents** to rewrite expressions
- relate the division property of exponents to looking *back* in time

You have seen how to multiply expressions with exponents. In this lesson you'll learn how to divide expressions with exponents.

Investigation: The Division Property of Exponents

Steps 1–3 For the expressions in Step 1 in your book, first write the numerators and denominators in expanded form and then eliminate factors equivalent to 1.

a. $\dfrac{5^9}{5^6} = \dfrac{\cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot 5 \cdot 5 \cdot 5}{\cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5}} = 5^3$

b. $\dfrac{3^3 \cdot 5^3}{3 \cdot 5^2} = \dfrac{\cancel{3} \cdot 3 \cdot 3 \cdot \cancel{5} \cdot \cancel{5} \cdot 5}{\cancel{3} \cdot \cancel{5} \cdot \cancel{5}} = 3^2 \cdot 5^1$

c. $\dfrac{4^4 x^6}{4^2 x^3} = \dfrac{\cancel{4} \cdot \cancel{4} \cdot 4 \cdot 4 \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{4} \cdot \cancel{4} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = 4^2 x^3$

Compare the exponents in each final expression to the exponents in the original quotient. Notice that, for each base, the exponent in the final expression is the exponent in the numerator minus the exponent in the denominator. You can use this idea to rewrite the expression in Step 3.

$$\frac{5^{15}\left(1 + \dfrac{0.08}{12}\right)^{24}}{5^{11}\left(1 + \dfrac{0.08}{12}\right)^{18}} = 5^{15-11}\left(1 + \frac{0.08}{12}\right)^{24-18} = 5^4\left(1 + \frac{0.08}{12}\right)^6$$

Steps 4–5 Exponential growth is related to repeated multiplication. When you look ahead in time, you multiply by more constant multipliers. To look back in time, you need to undo some of the multiplication, or divide. Complete parts a–d of Step 4. Here are the answers.

a. If $500(1 + 0.04)^7$ represents the balance after 7 years, then $\frac{500(1 + 0.04)^7}{(1 + 0.04)^3}$ represents the balance 3 years earlier (that is, after 4 years). You can rewrite this expression as $500(1 + 0.04)^4$.

b. If $21,300(1 - 0.12)^9$ represents the value after 9 years, then $\frac{21,300(1 - 0.12)^9}{(1 - 0.12)^5}$ represents the value 5 years earlier (that is, after 4 years). You can rewrite this expression as $21,300(1 - 0.12)^4$.

c. If the population after 5 weeks is $32(1 + 0.50)^5$, then the population 2 weeks earlier was $\frac{32(1 + 0.05)^5}{(1 + 0.50)^2}$ or $32(1 + 0.50)^3$.

d. If $A(1 + r)^n$ models n time periods of exponential growth, then $A(1 + r)^{n-m}$ models the growth m time periods earlier.

In general, to look back m time periods with an exponential growth model, divide by $(1 + r)^m$, where r is the rate of growth, or subtract m from the exponent.

(continued)

Lesson 6.5 • Looking Back with Exponents (continued)

In the investigation, you explored the **division property of exponents.** Read the statement of the property in your book. Then, read the examples on pages 361 to 363. Here are some more examples.

EXAMPLE A   Rewrite each expression with no denominator.

a. $\dfrac{p^7 q^5 r^3}{p^5 q^3 r}$

b. $\dfrac{5^2 \cdot 2^x \cdot 5^3}{2^y \cdot 5^4}$

▶ **Solution**

a. $\dfrac{p^7 q^5 r^3}{p^5 q^3 r} = p^{7-5} q^{5-3} r^{3-1} = p^2 q^2 r^2$

b. $\dfrac{5^2 \cdot 2^x \cdot 5^3}{2^y \cdot 5^4} = \dfrac{2^x \cdot 5^{2+3}}{2^y \cdot 5^4} = \dfrac{2^x \cdot 5^5}{2^y \cdot 5^4} = 2^{x-y} \cdot 5^{5-4} = 2^{x-y} \cdot 5$

EXAMPLE B   Eight hours ago there were 120 bacteria in a petri dish. Since then the population has increased by 75% each hour.

a. How many bacteria are in the population now?

b. How many bacteria were in the population 5 hours ago?

▶ **Solution**

a. The population has been increasing for 8 hours. The original population was 120, and the rate of growth is 0.75.

$A(1 + r)^x = 120(1 + 0.75)^8 \approx 10{,}556$

The current population is about 10,556 bacteria.

b. Five hours ago, the population had been growing for 3 hours.

$120(1 + 0.75)^3 \approx 643$

Five hours ago, the bacteria population was about 643.

Zero and Negative Exponents

In this lesson you will

- explore the meaning of **zero and negative exponents**
- rewrite expressions involving negative exponents
- write very small numbers in **scientific notation**

All the exponents you have worked with so far have been positive integers. In this lesson you will explore the meaning of zero and negative-integer exponents.

Investigation: More Exponents

Steps 1–2 Use the division property of exponents to rewrite each of the expressions in Step 1 in your book so that the result has a single exponent.

a. $\dfrac{y^7}{y^2} = y^5$ **b.** $\dfrac{3^2}{3^4} = 3^{-2}$ **c.** $\dfrac{7^4}{7^4} = 7^0$ **d.** $\dfrac{2}{2^5} = 2^{-4}$ **e.** $\dfrac{x^3}{x^6} = x^{-3}$

f. $\dfrac{z^8}{z} = z^7$ **g.** $\dfrac{2^3}{2^3} = 2^0$ **h.** $\dfrac{x^5}{x^5} = x^0$ **i.** $\dfrac{m^6}{m^3} = m^3$ **j.** $\dfrac{5^3}{5^5} = 5^{-2}$

When the exponent in the numerator is greater than the exponent in the denominator, the result has a positive exponent. When the exponent in the numerator is less than the exponent in the denominator, the result has a negative exponent. When the exponents in the numerator and the denominator are equal, the result has a zero exponent.

Steps 3–6 Look back at all the expressions in Step 1 that gave a result with a negative exponent. You can rewrite these expressions in a different way by first writing them in expanded form and then simplifying.

b. $\dfrac{3^2}{3^4} = \dfrac{\cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3} = \dfrac{1}{3^2}$ **d.** $\dfrac{2}{2^5} = \dfrac{\cancel{2}}{\cancel{2} \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \dfrac{1}{2^4}$

e. $\dfrac{x^3}{x^6} = \dfrac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x} = \dfrac{1}{x^3}$ **j.** $\dfrac{5^3}{5^5} = \dfrac{\cancel{5} \cdot \cancel{5} \cdot \cancel{5}}{\cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot 5 \cdot 5} = \dfrac{1}{5^2}$

Compare these results with the results from Step 1. Notice that a base raised to a negative exponent is the same as 1 over the same base raised to the opposite of that exponent.

Now, look back at the expressions in Step 1 that gave a result with a zero exponent. You can rewrite these expressions in a different way by expanding and simplifying.

a. $\dfrac{7^4}{7^4} = \dfrac{\cancel{7} \cdot \cancel{7} \cdot \cancel{7} \cdot \cancel{7}}{\cancel{7} \cdot \cancel{7} \cdot \cancel{7} \cdot \cancel{7}} = \dfrac{1}{1} = 1$ **g.** $\dfrac{2^3}{2^3} = \dfrac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} = \dfrac{1}{1} = 1$

h. $\dfrac{x^5}{x^5} = \dfrac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \dfrac{1}{1} = 1$

So a base raised to an exponent of zero is equal to 1.

(continued)

Lesson 6.6 • Zero and Negative Exponents (continued)

Steps 7–8 Use what you have learned to rewrite each expression in Step 7 in your book so that it has only positive exponents and only one fraction bar.

a. $\dfrac{5^{-2}}{1} = 5^{-2} = \dfrac{1}{5^2}$

b. $\dfrac{1}{3^{-8}} = \dfrac{1}{\frac{1}{3^8}} = \dfrac{3^8}{1} = 3^8$

c. $\dfrac{4x^{-2}}{z^2 y^{-5}} = \dfrac{\frac{4}{x^2}}{\frac{z^2}{y^5}} = \dfrac{4}{x^2} \cdot \dfrac{y^5}{z^2} = \dfrac{4y^5}{x^2 z^2}$

As a shortcut, you can rewrite fractions like those above by moving expressions involving exponents from the numerator to the denominator or vice versa, as long as you change the sign of the exponent with each move.

In your book, read the text and examples on pages 367 to 369. Example A gives you more practice simplifying expressions involving exponents. Example B shows how you can use negative exponents to look back in time with exponential growth situations. Example C shows how scientific notation can be used to write very small numbers. Below is an additional example involving scientific notation.

EXAMPLE Convert each number from standard notation to scientific notation or vice versa.

  a. An angstrom is a tiny unit of length equal to about 0.000000003973 inch.

  b. A proton has a mass of about 1.67×10^{-24} grams.

▶ **Solution** **a.** $0.000000003973 = 3.973 \times 0.000000001$

$$= 3.973 \times \frac{1}{1,000,000,000}$$

$$= 3.973 \times \frac{1}{10^9}$$

$$= 3.973 \times 10^{-9}$$

In general, to rewrite a number less than 1 in scientific notation, count the number of places you must move the decimal to the right to get a number between 1 and 10. Use the negative of that number as the exponent of 10.

b. $1.67 \times 10^{-24} = \dfrac{1.67}{10^{24}}$

$$= \frac{1.67}{1,000,000,000,000,000,000,000,000}$$

$$= 0.00000000000000000000000167$$

In general, when you are given a number in scientific notation with a negative exponent, you can convert it to standard notation by moving the decimal point to the left the number of places indicated by the exponent.

Fitting Exponential Models to Data

In this lesson you will

- fit **exponential models** to data
- use exponential models to **make predictions**

In earlier chapters you wrote equations to model linear data. In this lesson you will write equations to model data that shows an exponential pattern of growth or decay.
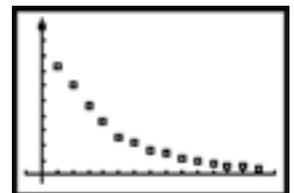
Investigation: Radioactive Decay

This investigation models the radioactive decay of a substance. Below, we work with sample data collected in one classroom. However, if you have the materials, it is a good idea to try collecting the data and completing the investigation on your own before reading the text below.

Steps 1–3   Read through Steps 1–3 in your book. Here is the data collected by one group. The last column will be discussed later.

"Years" elapsed	"Atoms" remaining	Successive ratios
0	201	
1	147	0.7313
2	120	0.8163
3	94	0.7833
4	71	0.7553
5	52	0.7324
6	42	0.8077
7	32	0.7619
8	28	0.8750
9	22	0.7857
10	18	0.8182
11	15	0.8333
12	12	0.8000
13	10	0.8333
14	9	0.9000

Steps 4–10   Here is a scatter plot of the data. Notice that the points appear to follow an exponential pattern.

To fit an exponential equation to this data, you need to find a number to use as the constant multiplier. To do this, first compute the ratios of successive "atoms remaining" values. The results are shown in the table. The ratios are fairly close. We'll use the mean, 0.802, as a representative ratio. Because about 0.802 or 80.2% of the atoms remain each year, about 100% − 80.2%, or 19.8%, of the atoms decay.
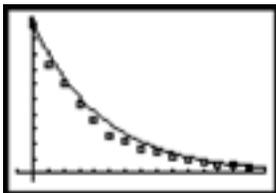

[0, 14, 1, 0, 200, 25]

(continued)

So to write an exponential equation to model this situation, we can use starting value 201 (the number of "atoms" at the start of the experiment) and decay rate 0.198. The equation is $y = 201(1 - 0.198)^x$. Here, this equation is graphed in the same window as the scatter plot.

The equation does not appear to fit the data very well. Adjust the values of A and r until you get a better fit.

Here is the graph of $y = 195(1 - 0.220)^x$. This equation appears to fit the data quite well.
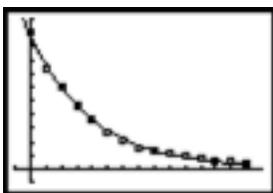
Steps 11–12 The group that collected the data in the table used a plate with a 68° angle. The section enclosed by the angle made up $\frac{68}{360}$ or 19% of the plate's area. This is close to the decay rate used in the equation model. This makes sense because if the counters land so that they are distributed evenly, about 19% of them will land in the 68° section—that is, 19% of the atoms will decay.

Create a calculator table for the model equation $y = 195(1 - 0.220)^x$. Compare the values in the calculator table to those in the table of data. Notice that although none of the values in the calculator table are exactly the same as the actual data values, most of them are very close.

Now, read the text and example that follow the investigation in your book.