CONDENSED LESSON

8.1 Translating Points

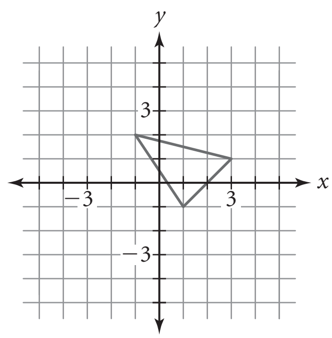
In this lesson you will

- **translate** figures on the coordinate plane
- define a **translation** by describing how it affects a general point (x, y)

A mathematical rule that changes or moves a figure is called a **transformation.** In this lesson, you explore one type of transformation.

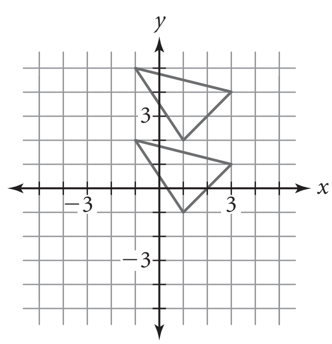
Investigation: Figures in Motion

Steps 1–6 The triangle below has vertices $(-1, 2)$, $(1, -1)$, and $(3, 1)$.

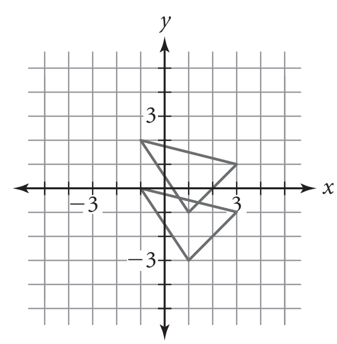


If you add 3 to each y-coordinate, you get $(-1, 5)$, $(1, 2)$, and $(3, 4)$. If you subtract 2 from each y-coordinate, you get $(-1, 0)$, $(1, -3)$, and $(3, -1)$. The grids below show the original triangle and triangles with vertices at the "transformed" points.

Add 3 to the y-coordinates



Subtract 2 from the y-coordinates



Notice that *adding* 3 to the y-coordinates moves the triangle *up* 3 units and that *subtracting* 2 from the y-coordinates moves the triangle *down* 2 units.

Now, draw your own triangle, and move it by adding or subtracting a number from the y-coordinates of the vertices.

Each grid in Step 6 on page 438 shows an original triangle and the triangle that results from adding a number to or subtracting a number from the y-coordinates. See if you can figure out what number was added or subtracted.

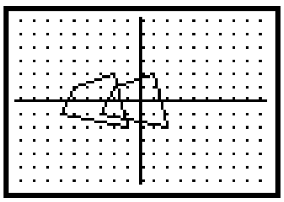
(continued)

Here are the answers to Step 6.

a. 3 was added. **b.** 4 was subtracted. **c.** 5 was subtracted.

Steps 7–13 Now, you will draw and move a polygon by using your calculator. The vertices of the quadrilateral shown in Step 7 are $(1, 2)$, $(2, -2)$, $(-3, -1)$, and $(-2, 1)$. Follow Step 8 to enter the coordinates and draw the quadrilateral on your calculator.

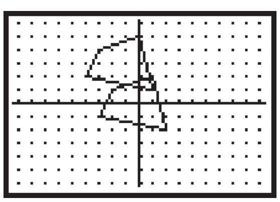
Define lists L_3 and L_4 so that $L_3 = L_1 - 3$ and $L_4 = L_2$. So, L_3 contains the original x-coordinates minus 3, and L_4 contains the original y-coordinates. Graph a new quadrilateral using L_3 for the x-coordinates and L_4 for the y-coordinates.



The vertices of the new quadrilateral are $(-2, 2)$, $(-1, -2)$, $(-6, -1)$, and $(-5, 1)$. Notice that subtracting 3 from the x-coordinates of the original quadrilateral shifts it left 3 units.

Follow Steps 9 and 10 at least two more times, adding a different number to or subtracting a different number from the x-coordinates each time. You should find that adding a positive number to the x-coordinates shifts the figure right that many units and that subtracting a positive number from the x-coordinates shifts the figure left that many units.

Now, let $L_3 = L_1 - 1$ and let $L_4 = L_2 + 3$. This subtracts 1 from the original x-coordinates and adds 3 to the original y-coordinates, shifting the quadrilateral left 1 unit and up 3 units, as shown below.



Each graphing window in Step 12 shows the original quadrilateral and a new, "transformed" quadrilateral. Write definitions for L_3 and L_4 in terms of L_1 and L_2 that would create the transformed quadrilateral. Here are the correct rules.

a. $L_3 = L_1 + 6, L_4 = L_2$ **b.** $L_3 = L_1 - 3, L_4 = L_2$ **c.** $L_3 = L_1 - 5, L_4 = L_2 + 3$

When you transform a figure, the result is called the **image** of the original figure. Horizontal and vertical transformations, like the ones you explored in the investigation, are called **translations.** You can define a translation by describing the image of a general point (x, y). For example, the translation that shifts a figure left 4 units and up 2 units can be defined as $(x - 4, y + 2)$.

Now, read the example in your book.

CONDENSED LESSON

8.2 Translating Graphs

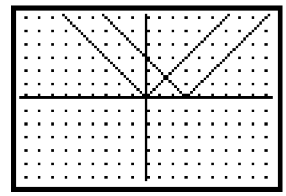
In this lesson you will

- **translate** the **absolute-value** and **squaring functions**
- **translate** an **exponential function**
- learn about **families of functions**

In the previous lesson you translated figures on the coordinate plane. In this lesson you will learn how to translate functions.

Investigation: Translations of Functions

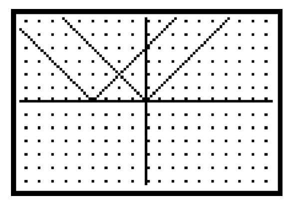
Steps 1–6 If you substitute $x - 3$ for x in the absolute-value function $y = |x|$, you get $y = |x - 3|$. You can think of this as finding $f(x - 3)$ when $f(x) = |x|$. Enter $y = |x|$ into Y1 and $y = |x - 3|$ into Y2, and graph both functions.



Notice that the graph of $y = |x - 3|$ is the graph of $y = |x|$ translated right 3 units.

The **vertex** of an absolute value graph is the point where the function changes from decreasing to increasing or from increasing to decreasing. The vertex of $y = |x|$ is $(0, 0)$, and the vertex of $y = |x - 3|$ is $(3, 0)$. So, the vertex of $y = |x|$, like the rest of the graph, has been translated right 3 units.

The function $y = |x - (-4)|$ or $y = |x + 4|$ translates the graph of $y = |x|$ left 4 units. To get $y = |x + 4|$, you substitute $x + 4$ for x in $y = |x|$.

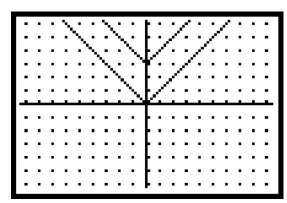


Write a function to create each translation of $y = |x|$ shown in Step 6. Use your calculator to check your work. You should get these results.

a. $y = |x - 2|$ **b.** $y = |x - 5|$ **c.** $y = |x + 3|$

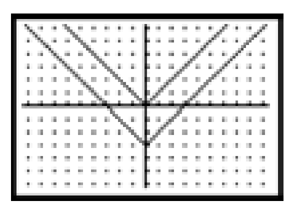
Steps 7–12 Now, you will translate $y = |x|$ along the y-axis.

If you substitute $y - 3$ for y in $y = |x|$, you get $y - 3 = |x|$ or (solving for y) $y = 3 + |x|$. Here are the graphs of $y = |x|$ and $y = 3 + |x|$ on the same axes.



Notice that the graph of $y = 3 + |x|$ is the graph of $y = |x|$ translated up 3 units. The vertex of $y = 3 + |x|$ is $(0, 3)$, which is the vertex of $y = |x|$ translated up 3 units.

If you substitute $y - (-3)$ or $y + 3$ for y in $y = |x|$, you get $y + 3 = |x|$ or $y = -3 + |x|$. The graph of $y = -3 + |x|$ is the graph of $y = |x|$ translated down 3 units.



Write a function to create each translation of $y = |x|$ shown in Step 12. Use your calculator to check your work. You should get these results.

a. $y = -2 + |x|$ **b.** $y = 1 + |x|$ **c.** $y = -4 + |x - 3|$

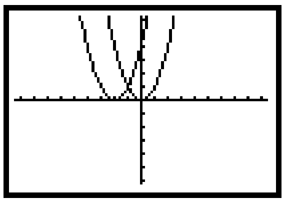
(continued)

Step 13 You have seen that to translate the graph of $y = |x|$ horizontally, you subtract a number from x in the function. Subtracting a positive number translates the graph right, and subtracting a negative number translates the graph left. To translate the graph vertically, you add a number to the entire function. Adding a positive number translates the graph up, and adding a negative number translates the graph down.

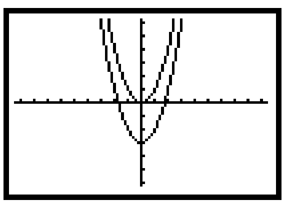
EXAMPLE

The same ideas you used to translate the absolute-value function can be used to translate the function $y = x^2$.

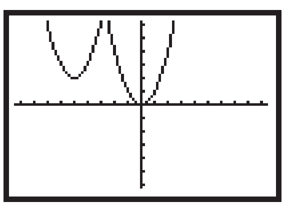
Here is the graph of $y = x^2$ and $y = (x + 2)^2$.



Here is the graph of $y = x^2$ and $y = x^2 - 3$.



Here is the graph of $y = x^2$ and $y = (x + 5)^2 + 2$.



The vertex of a parabola is the point where the graph changes from decreasing to increasing or from increasing to decreasing. The vertices of the translated parabolas above are $(-2, 0)$, $(0, -3)$, and $(-5, 2)$. Notice that the x-coordinate of the vertex is the value subtracted from x in the function and that the y-coordinate is the value added to the entire function.

The functions $y = |x|$ and $y = x^2$ are examples of **parent functions.** By transforming a parent function, you can create infinitely many functions in the same **family of functions.** For example, the functions $y = (x + 2)^2$ and $y = x^2 - 4$ are both part of the squaring family of functions, which has $y = x^2$ as the parent function.

Now, read Example B in your book, which shows you how to translate an exponential function.

CONDENSED
LESSON
8.3 Reflecting Points and Graphs

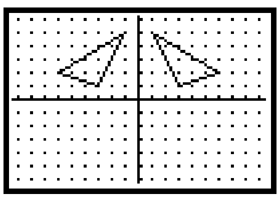
In this lesson you will

- **reflect polygons** over the x- and y-axes
- **reflect graphs of functions** over the x- and y-axes
- write equations for graphs created by **combining transformations**

You have studied translations—transformations that slide a figure horizontally or vertically. In this lesson you will learn about transformations that flip a figure across a line.

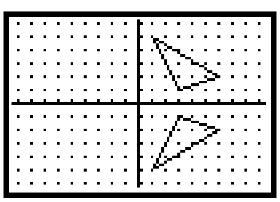
Investigation: Flipping Graphs

Steps 1–5 The triangle on page 453 of your book has vertices (1, 5), (3, 1), and (6, 2). The grid below shows the original triangle and the triangle formed by changing the sign of the x-coordinate of each vertex to get $(-1, 5)$, $(-3, 1)$, and $(-6, 2)$.



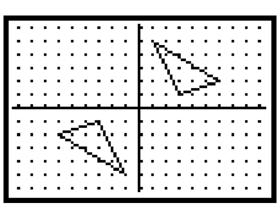
Changing the signs of the x-coordinates *flips* a figure across the y-axis, creating a mirror image of the original. You can match the original figure exactly with the image by folding the grid along the y-axis.

The grid below shows the original triangle and the triangle formed by changing the sign of the y-coordinate of each vertex to get $(1, -5)$, $(3, -1)$, and $(6, -2)$.



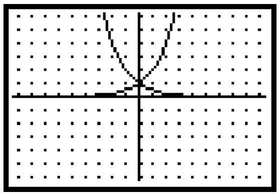
Changing the signs of the y-coordinates *flips* a figure across the x-axis, creating a mirror image of the original. You can match the original figure exactly with the image by folding the grid along the x-axis.

You change the signs of both the x- and y-coordinates to get $(-1, -5)$, $(-3, -1)$, and $(-6, -2)$. This transformation flips the figure across one axis and then across the other.

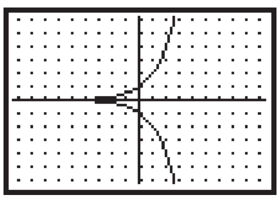


(continued)

Steps 6–10 If you replace the x in the function $y = 2^x$ with $-x$, you get $y = 2^{-x}$. (This is the same as finding $f(-x)$ when $f(x) = 2^x$.) If you graph both functions on your calculator, you'll see that the graph of $y = 2^{-x}$ is the graph of $y = 2^x$ flipped across the y-axis.



If you replace the y in $y = 2^x$ with $-y$, you get $-y = 2^x$ or (solving for y) $y = -2^x$. (This is the same as finding $-f(x)$ when $f(x) = 2^x$.) If you graph both functions on your calculator, you'll see that the graph of $y = -2^x$ is the graph of $y = 2^x$ flipped across the x-axis.



For each function in Step 9, replace x with $-x$, and graph both the original function and the new function. In each case you should find that the graph of the original function is flipped across the y-axis to get the graph of the new function. (Note: Because the graph of $y = |x|$ is symmetric across the y-axis, it looks the same when it is flipped across the y-axis. So, the graphs of $y = |x|$ and $y = |-x|$ are identical.)

For each function in Step 9, replace y with $-y$, and graph both the original function and the new function. In each case you should find that the graph of the original function is flipped across the x-axis to get the graph of the new function.

A transformation that flips a figure to create a mirror image is called a **reflection.** As you discovered in the investigation, a point is **reflected across the *x*-axis** when you change the sign of its y-coordinate. A point is **reflected across the *y*-axis** when you change the sign of its x-coordinate. Similarly, a function is reflected across the x-axis when you change the sign of y, and a function is reflected across the y-axis when you change the sign of x.

Read the rest of the lesson in your book. Read the examples very carefully. Example B explains how to write equations for graphs created by applying more than one transformation to the graph of a parent function.

CONDENSED LESSON

8.4 Stretching and Shrinking Graphs

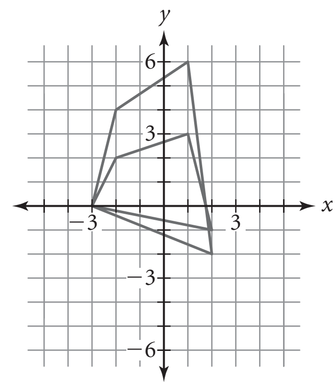
In this lesson you will

- **stretch** and **shrink** a **quadrilateral**
- **stretch** and **shrink** the graph of a **function**
- write equations for graphs formed by **combining transformations**

You have learned about transformations that slide a figure horizontally or vertically and that flip a figure across a line. In this lesson you'll study a transformation that changes a figure's shape.

Investigation: Changing the Shape of a Graph

Steps 1–6 Copy the quadrilateral on page 463 of your book onto graph paper, or enter the x-coordinates into list L1 and the y-coordinates into list L2. The coordinates of the vertices are $(1, 3)$, $(2, -1)$, $(-3, 0)$, and $(-2, 2)$.

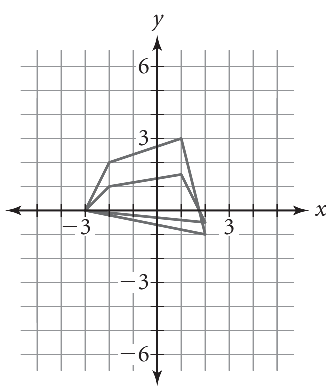


Multiply the y-coordinate of each vertex by 2 to get $(1, 6)$, $(2, -2)$, $(-3, 0)$, and $(-2, 4)$. On the same grid as the original figure, draw a new quadrilateral with these new points as vertices. Or follow the instructions in your book to graph it on your calculator.

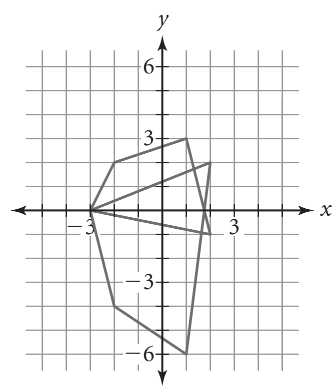
As you can see, multiplying the y-coordinates by 2 stretches the figure vertically. Points above the x-axis move up. Points below the x-axis move down. Points on the x-axis remain fixed.

Now, multiply the y-coordinates of the vertices of the original quadrilateral by 3, by 0.5, and by -2, and draw the resulting quadrilaterals. Below are the results for 0.5 and -2.

Multiply by 0.5



Multiply by −2



Multiplying by 0.5 shrinks the figure vertically to half its size. Multiplying by -2 stretches the figure vertically and flips it over the x-axis. In general, multiplying the y-coordinates of a figure by a number a

- stretches the figure if $|a| > 1$ and shrinks the figure if $|a| < 1$
- reflects the figure across the x-axis if $a < 0$

(continued)

Steps 7–9 Graph the triangle shown in Step 7 on your calculator, putting the x-coordinates in list L_1 and the y-coordinates in list L_2. Then, predict how the definitions in parts a and b of Step 8 will transform the triangle, and use your calculator to check your answers. Here are the results.

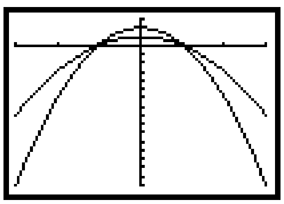
a. The figure shrinks vertically to half its size and is flipped over the x-axis.

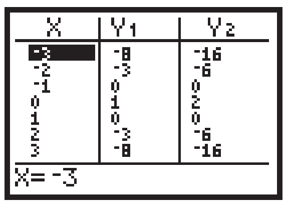
b. The figure stretches vertically to twice its size and is translated down 2 units.

In Step 9, write definitions for lists L_3 and L_4 that would create each image. Here are the answers.

a. $L_3 = L_1$; $L_4 = 3 \cdot L_2$ **b.** $L_3 = L_1$; $L_4 = 2 \cdot L_2 + 3$

Steps 10–14 Enter the equation $f(x) = -x^2 + 1$ as Y_1, and graph it on your calculator. Multiply the right side of the equation by 2—that is, find $y = 2 \cdot f(x)$—and enter the result, $y = 2(-x^2 + 1)$, as Y_2. Below are the graph and table for the two functions.





X	Y1	Y2
-3	-8	-16
-2	-3	-6
-1	0	0
0	1	2
1	0	0
2	-3	-6
3	-8	-16

X= -3

[−3, 3, 1, −16, 3, 1]

In the table, notice that each y-value for $y = 2(-x^2 + 1)$ is twice the corresponding y-value for $y = -x^2 + 1$. This causes the graph to stretch. The points on the graph of $y = 2(-x^2 + 1)$ are twice as far from the x-axis as the corresponding points on the graph of $y = -x^2 + 1$. Multiplying by 2 causes the points above the x-axis to move up and the points below the x-axis to move down.

Repeat the process above for $y = 0.5 \cdot f(x)$, $y = 3 \cdot f(x)$, and $y = -2 \cdot f(x)$. You should find the following:

- Multiplying by 0.5 gives y-values that are half the corresponding y-values for the original function, resulting in a vertical shrink.
- Multiplying by 3 gives y-values that are 3 times the corresponding y-values for the original function, resulting in a vertical stretch.
- Multiplying by -2 gives y-values that are -2 times the corresponding y-values for the original function, resulting in a vertical stretch and a reflection across the x-axis.

Look at the graphs in Step 14. Use what you have learned about stretching and shrinking graphs to write an equation for $R(x)$ in terms of $B(x)$ and an equation for $B(x)$ in terms of $R(x)$. You should get the following results:

a. $R(x) = 3 \cdot B(x)$; $B(x) = \frac{1}{3} \cdot R(x)$

b. $R(x) = -\frac{1}{2} \cdot B(x)$; $B(x) = -2 \cdot R(x)$

Read the rest of the lesson, including the examples, very carefully.

CONDENSED LESSON

8.6 Introduction to Rational Functions

In this lesson you will

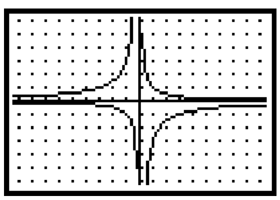
- explore transformations of the parent function $y = \frac{1}{x}$

In Chapter 2, you learned about inverse variation. The simplest inverse variation equation is $y = \frac{1}{x}$. On page 474 of your book, read about $y = \frac{1}{x}$ and its graph. In this lesson you will see how the parent function $y = \frac{1}{x}$ can help you understand many other functions.

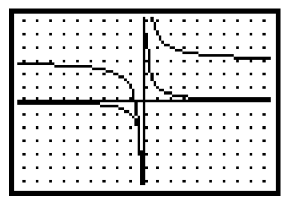
Investigation: I'm Trying to Be Rational

Steps 1–5 Graph the parent function $y = \frac{1}{x}$ on your calculator. The functions in Step 2 are all transformations of $y = \frac{1}{x}$. Use what you have learned in this chapter to predict how the graph of each function will compare to the graph of $y = \frac{1}{x}$. The answers are given below.

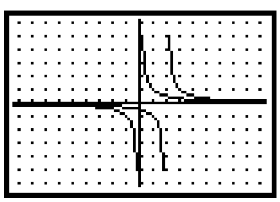
a. Vertical stretch by a factor of 3 and reflection across the x-axis



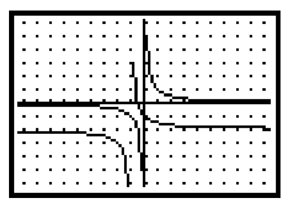
b. Vertical stretch by a factor of 2 and translation up 3 units



c. Translation right 2 units



d. Translation left 1 unit and down 2 units



Now, without graphing, describe what the graph of each function in Step 4 looks like. Use the words *linear, nonlinear, increasing,* and *decreasing*. Define the domain and range, and give equations for the asymptotes. Here are sample answers.

Graph a is a vertical stretch of $y = \frac{1}{x}$ by a factor of 5, followed by a translation right 4 units; it is nonlinear and decreasing; the domain is $x \neq 4$, and the range is $y \neq 0$; there are asymptotes at $x = 4$ and $y = 0$.

Graph b is a reflection of $y = \frac{1}{x}$ across the x-axis, followed by a translation left 3 units and down 5 units; it is nonlinear and increasing; the domain is $x \neq -3$, and the range is $y \neq -5$; there are asymptotes at $x = -3$ and $y = -5$.

Graph c is a vertical stretch of $y = \frac{1}{x}$ by a factor of $|a|$ (if $a < 0$, the graph is also reflected across the x-axis), followed by a translation horizontally h units and vertically k units; it is nonlinear and decreasing if a is positive and nonlinear and increasing if a is negative; the domain is $x \neq h$, and the range is $y \neq k$; there are asymptotes at $x = h$ and $y = k$.

(continued)

Functions such as $y = \frac{5}{x-4}$ are called **rational functions** because they involve ratios of two expressions. Not all rational functions are transformations of $y = \frac{1}{x}$, but the graph of any rational function shares some similarities with the graph of $y = \frac{1}{x}$.

Step 6 The parent function $y = \frac{1}{x}$ has asymptotes at $x = 0$ and $y = 0$. These must be translated left 2 units and up 1 unit. So, one possible function is $y - 1 = \frac{1}{x+2}$, or $y = \frac{1}{x+2} + 1$.

In general, the following is true if the parent function $y = \frac{1}{x}$ is changed to $y - k = \frac{a}{x-h}$ or $y = \frac{a}{x-h} + k$:

- The graph of $y = \frac{1}{x}$ is stretched by a factor of $|a|$ (and reflected over the x-axis if $a < 0$) and translated right h units and up k units.
- There is a vertical asymptote at $x = h$ and a horizontal asymptote at $y = k$.
- The domain is $x \neq h$ and the range is $y \neq k$.

Now, read the examples in your book. Example A uses a rational function to model a real-world situation. Example B shows how to perform arithmetic operations with rational expressions.

CONDENSED LESSON

8.7 Transformations with Matrices

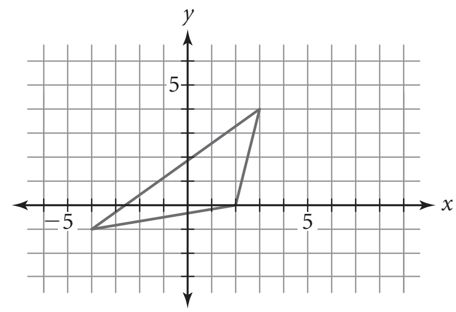
In this lesson you will

- relate translations to **matrix addition**
- relate reflections, stretches, and shrinks to **matrix multiplication**

In your book, read the text before the investigation, which explains how to use a matrix to represent the vertices of a geometric figure.

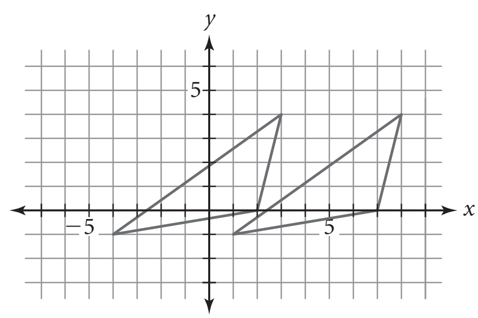
Investigation: Matrix Transformations

Steps 1–6 Matrix $[A] = \begin{bmatrix} -4 & 3 & 2 \\ -1 & 4 & 0 \end{bmatrix}$ represents a triangle with vertices $(-4, -1)$, $(3, 4)$, and $(2, 0)$.



Adding the matrix $\begin{bmatrix} 5 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ to matrix $[A]$ adds 5 to each x-coordinate, which translates the triangle right 5 units:

$$[A] + \begin{bmatrix} 5 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 2 \\ -1 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 7 \\ -1 & 4 & 0 \end{bmatrix}$$



In parts a–c of Step 5, find the matrix sum and graph the resulting triangle. You should find that the sums correspond to the following translations of the original triangle:

a. down 4 units **b.** right 5 units, down 4 units **c.** left 6 units, up 4 units

Now, write matrix equations to represent the translations in Step 6. Here are the answers.

a. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1.5 & 1.5 & 1.5 & 1.5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 1.5 & 2.5 & 2.5 & 1.5 \end{bmatrix}$

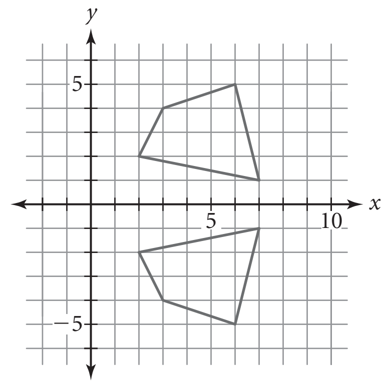
b. $\begin{bmatrix} -3 & -2 & 1 & 2 \\ -1 & 1 & 2 & -2 \end{bmatrix} + \begin{bmatrix} -10.4 & -10.4 & -10.4 & -10.4 \\ 6.9 & 6.9 & 6.9 & 6.9 \end{bmatrix} = \begin{bmatrix} -13.4 & -12.4 & -9.4 & -8.4 \\ 5.9 & 7.9 & 8.9 & 4.9 \end{bmatrix}$

(continued)

Steps 7–11 Copy the quadrilateral on page 485 of your book onto graph paper. Matrix $[B] = \begin{bmatrix} 2 & 3 & 6 & 7 & x \\ 2 & 4 & 5 & 1 & y \end{bmatrix}$ gives the coordinates of the vertices, along with the coordinates of a general point, (x, y).

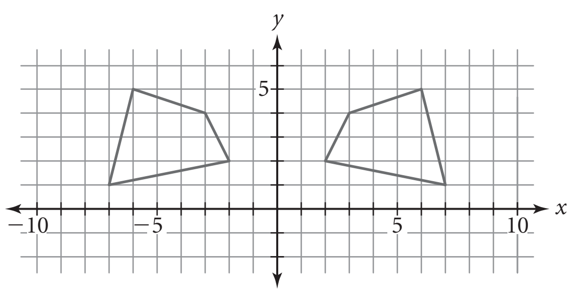
Calculating $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot [B]$ multiplies each y-coordinate by -1, which reflects the quadrilateral across the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 6 & 7 & x \\ 2 & 4 & 5 & 1 & y \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 & 7 & x \\ -2 & -4 & -5 & -1 & -y \end{bmatrix}$$



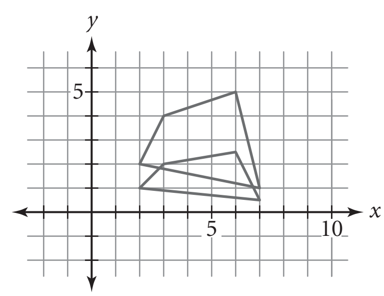
Now, perform the multiplications in Step 11 and graph the resulting quadrilaterals. You should get the following results:

a. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 6 & 7 & x \\ 2 & 4 & 5 & 1 & y \end{bmatrix} = \begin{bmatrix} -2 & -3 & -6 & -7 & -x \\ 2 & 4 & 5 & 1 & y \end{bmatrix}$. The x-coordinates are multiplied by -1. This reflects the quadrilateral across the y-axis.



b. $\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 6 & 7 & x \\ 2 & 4 & 5 & 1 & y \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 & 7 & x \\ 1 & 2 & 2.5 & 0.5 & 0.5y \end{bmatrix}$.

The y-coordinates are multiplied by 0.5. This shrinks the quadrilateral vertically by a factor of 0.5.



c. $\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 6 & 7 & x \\ 2 & 4 & 5 & 1 & y \end{bmatrix} = \begin{bmatrix} 1 & 1.5 & 3 & 3.5 & 0.5x \\ 4 & 8 & 10 & 2 & 2y \end{bmatrix}$.

The x-coordinates are multiplied by 0.5, and the y-coordinates are multiplied by 2. This shrinks the quadrilateral horizontally by a factor of 0.5 and stretches it vertically by a factor of 2.